Advanced Mechanics

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Notes adapted from Mark Lusk (Colorado School of Mines), Sebastian Streichan (UCSB), Omer Blaes (UCSB), and Tengiz Bibilashvili (UCSB). Main text: Taylor's Classical Mechanics.

Goal of classical mechanics: determine equations of motion for objects in physical settings (ball on a hill, particle in a box, bead on a wire, ball in an elevator on a rotating planet orbiting a star)

You should already be familiar with Newtonian mechanics:

$\sum_{i=1}^{N} \vec{F}_i = m\vec{a} = m\ddot{\vec{r}}(t)$ ... differential equation to be solved for $\vec{r}(t)$

3 degrees of freedom $(\hat{x}, \hat{y}, \hat{z}) \Rightarrow$ 3N coupled ODEs

① Newtonian mechanics are local: objects "react" to forces.

② Newtonian mechanics give rise to conserved quantities
(e.g. $\vec{F}_{ext} = \dot{\vec{p}} \Rightarrow$ in absence of external forces, momentum is conserved; (Taylor 1.24)
$\dot{\vec{L}}_{ext} = \vec{\Gamma}_{ext} \Rightarrow$ in absence of external torques, angular momentum conserved; (T 3.26)
$\Delta T = \int \vec{F} \cdot d\vec{r} \Rightarrow$ in absence of external work, kinetic energy is conserved) (T 4.7)

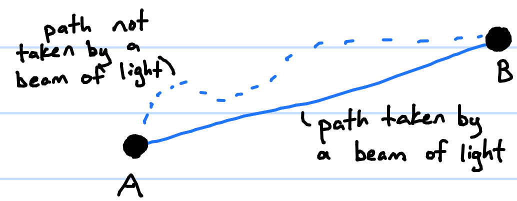
③ Newtonian mechanics explicitly account for constraint forces (e.g. normal forces or tensions)

In this course we will develop the more advanced approaches of Lagrangian and Hamiltonian mechanics. Ultimately, these approaches are equivalent to Newtonian mechanics, but they conveniently:

① are independent of coordinate system

② automatically take constraint forces into account

③ provide global statements about the motion of objects. e.g., Fermat's principle states that the path light takes is the path that is traversed most quickly:



Mathematically, Fermat's principle may be phrased as finding the path that minimizes $\int_{A \to B} dt$.
Roughly speaking, local $\Leftrightarrow$ derivative and global $\Leftrightarrow$ integral.

To solve such global problems, we need to equip ourselves with variational calculus aka the calculus of variations, which will enable us to answer:

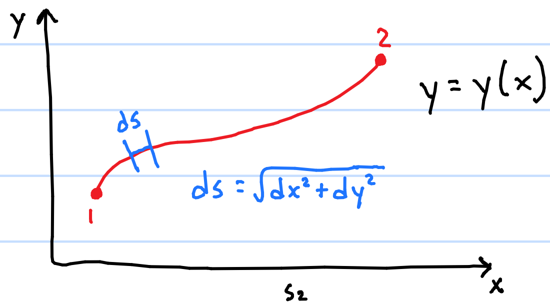
ex) the shortest path between two points (on or off a sphere)

ex) the shape of a hanging cable (suspension bridge)

ex) the shape of a roller coaster that will drop you to the ground the most quickly

Chapter 6: Calculus of variations

Goal: Find shortest path between 2 points



Length of path, $L = \int_{s_1}^{s_2} ds$

But $ds = \sqrt{dx^2+dy^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx = \sqrt{1+(y'(x))^2}\, dx$

$$\Rightarrow L = \int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}\, dx$$

Want to find $y(x)$ that minimizes L.

More generally, we often want to minimize expressions like $A[y]$:

$$A[y] = \int_{x_1}^{x_2} dx\, f(y(x), y'(x), x)$$

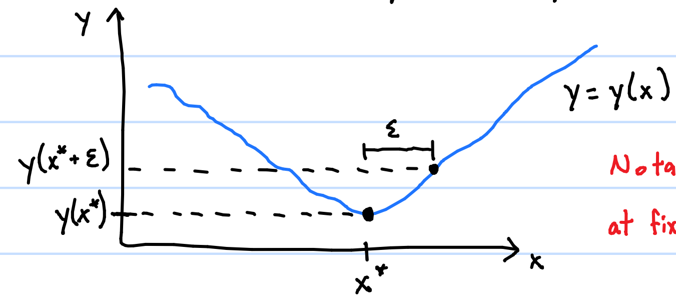
Here, A is a functional of y.

To minimize A, we want something like "$\frac{dA}{dy} = 0$."

But how do you take the derivative of a functional?

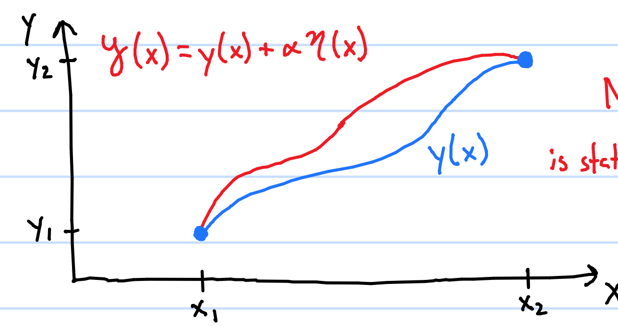
(in mathematics, this is formally a Gateaux derivative)

In traditional calculus, a function $y(x)$ is minimized at a point x^* when $y(x^*) < y(x^*+\varepsilon)$ for $|\varepsilon|$ small:



Notation: at fixed points, $\frac{dy}{dx} = 0$

The analagous concept in variational calculus is that a functional $A[y]$ is minimized for a curve $y(x)$ when $A[y(x)] \leq A[y(x) + \alpha\eta(x)]$ for any function $\eta(x)$ and small $|\alpha|$ (with $\eta(x_1) = \eta(x_2) = 0$)



Notation: functional is stationary when $\delta y = 0$

"variation in y"

$y(x)$ ... function that minimizes A

α ... any real number

$\eta(x)$ ... any function with $\eta(x_1) = \eta(x_2) = 0$

Now we can think of $A[y]$ as a function of α, written as $A[\alpha; y]$. Note A is minimized for $\alpha = 0$.

new "variational" notation: $\delta A \equiv$ "variation in A"

To minimize $A[\alpha; y]$ as a function of α, need $\frac{dA}{d\alpha} = 0$ "for all" $\forall\, \eta(x)$ at $\alpha = 0$ $\equiv$ $\delta A = 0$

"A is stationary when the variation in A vanishes"

$$A[\alpha; y] = \int_{x_1}^{x_2} dx\, f(\underbrace{y+\alpha\eta,\ y'+\alpha\eta',\ x}_{\equiv \vec{b}})$$

Leibniz integral rule

$$\Rightarrow \frac{dA}{d\alpha} = \int_{x_1}^{x_2} dx\, \frac{\partial f(y(x)+\alpha\eta(x),\ y'(x)+\alpha\eta'(x),\ x)}{\partial \alpha}$$

chain rule

$$= \int_{x_1}^{x_2} dx \left[\left(\frac{\partial f}{\partial y}\Big|_{\vec{b}} \right) \eta + \left(\frac{\partial f}{\partial y'}\Big|_{\vec{b}} \right) \eta' \right]$$

Rewrite second term using integration by parts:

$$\int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y'}\Big|_{\vec{b}} \eta' = \left[\frac{\partial f}{\partial y'}\Big|_{\vec{b}} \underset{\text{0 at } x_1 \text{ and } x_2}{\eta} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} dx\, \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\Big|_{\vec{b}} \right) \eta$$

Therefore,

$$\frac{dA}{d\alpha} = \int_{x_1}^{x_2} dx \underbrace{\left[\frac{\partial f}{\partial y}\Big|_{\vec{b}} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\Big|_{\vec{b}} \right) \right]}_{\equiv g(x)} \eta$$

We want $\frac{dA}{d\alpha} = 0\ \forall\, \eta$ at $\alpha = 0$. So we need $\int_{x_1}^{x_2} dx\, g(x)\, \eta(x) = 0$ for any choice of $\eta(x)$. This can only occur if $g(x)$ is identically 0 at $\alpha = 0$.

Thus we require

$$\frac{\partial f}{\partial y}\Big|_{\vec{b}} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\Big|_{\vec{b}} \right) = 0 \quad \text{at } \alpha = 0.$$

But $\vec{b} = \{ y(x) + \alpha\eta(x),\ y'(x) + \alpha\eta'(x),\ x \}$

$\Rightarrow \vec{b} = \{ y(x),\ y'(x),\ x \}$ at $\alpha = 0$

Therefore we have a necessary condition for minimizing $A[y]$:

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0}$$

Euler-Lagrange Eqns

$\Rightarrow$ We have a recipe for identifying the path $y(x)$ that minimizes a functional $A[y]$.

The broader point here is that global principles have local implications

This is powerful, since many problems in physics may be framed as "which path minimizes this quantity?"

Now we may use the E-L equations to find the shortest path between 2 points

$$L = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y'(x))^2}}_{f(y,y',x)}\,dx \;; \quad \text{E-L: } \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{2y'}{2\sqrt{1+y'^2}} = y'(1+y'^2)^{-1/2}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = y''(1+y'^2)^{-1/2} - \frac{1}{2}y'(1+y'^2)^{-3/2}\,2y'y''$$

Plugging these quantities into E-L,

$$y''\left[(1+y'^2)^{-3/2}(1+y'^2-y'^2)\right] = 0 \Rightarrow y''=0$$

$$\Rightarrow \boxed{y(x) = c_1 x + c_0}$$

straight line, w/ constants based on endpoints

Alternatively, we could leverage $\frac{\partial f}{\partial y} = 0$, and note $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \text{const}$

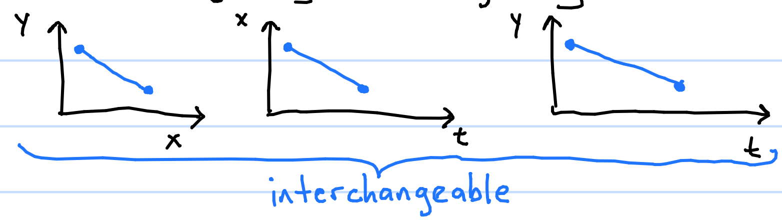
$$\Rightarrow y'(1+y'^2)^{-1/2} = c \Rightarrow y'^2 = c(1+y'^2)$$

$$\Rightarrow y'^2(1-c) = c \Rightarrow y' = \sqrt{\frac{c}{1-c}} = \text{some constant}$$

$$\Rightarrow y(x) = c_1 x + c_0$$

Many paths to the top of the mountain!

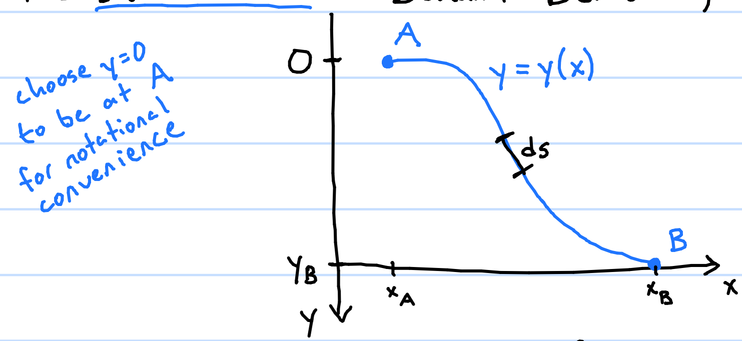
Naturally, any approach regarding a path $y(x)$ may be used regarding a trajectory $x(t)$ or $y(t)$



Next we turn to a classic CM problem, posed by Bernoulli and solved by Newton (1697):

ex) $\underbrace{\text{Brachisto}}_{\text{shortest}}\overbrace{\text{chrone}}^{\text{time}}$ problem (roller coaster):

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time" —Johann Bernoulli, 1696



Want to minimize the time $\int_A^B dt$, but distance = (time)(velocity) $\Rightarrow$ minimize $\int_A^B \frac{ds}{v}$, for velocity v.

Conservation of energy $\Rightarrow \frac{1}{2}mv^2 = mg\Delta y \Rightarrow v = \sqrt{2gy}$

Then, we want to minimize

$$T = \int_A^B \frac{ds}{\sqrt{2gy}} = \int_{x_1}^{x_2} \underbrace{\frac{dx\sqrt{1+(y'(x))^2}}{\sqrt{2gy(x)}}}_{f(y,y',x)} = \int_0^{y_B} \underbrace{\frac{dy\sqrt{1+(x'(y))^2}}{\sqrt{2gy}}}_{f(x,x',y)}$$

which to choose?

Choosing x versus y as the independent variable doesn't ultimately matter – we will proceed w/ x as the independent variable, in the HW you will repeat the problem using y.

$$T = \frac{1}{\sqrt{2g}}\int_0^{y_B} dy\sqrt{\frac{1+x'^2}{y}} \; ; \quad \text{E-L: } \frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y(1+x'^2)}} \; ; \quad \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0 \Rightarrow \frac{\partial f}{\partial x'} = \text{const}$$

$$\Rightarrow x'^2 = c\left(y(1+x'^2)\right) \quad \Rightarrow x' = \sqrt{\frac{cy}{1-cy}}$$

$$\Rightarrow x(y) = \int_0^y d\tilde{y}\sqrt{\frac{c\tilde{y}}{1-c\tilde{y}}}.$$ How to solve? Integral look-up tables or mathematica.

Choose $\tilde{y} = \frac{1}{2c}(1-\cos\theta) \Rightarrow d\tilde{y} = \frac{\sin\theta}{2c}\,d\theta$

$$\Rightarrow x(y) = \int_0^{\cos^{-1}(1-2cy)} \frac{d\theta}{2c}\sin\theta\sqrt{\frac{1-\cos\theta}{2-(1-\cos\theta)}} = \frac{1}{2c}\int_0^{\cos^{-1}(1-2cy)} d\theta\,\sin\theta\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

trig identity

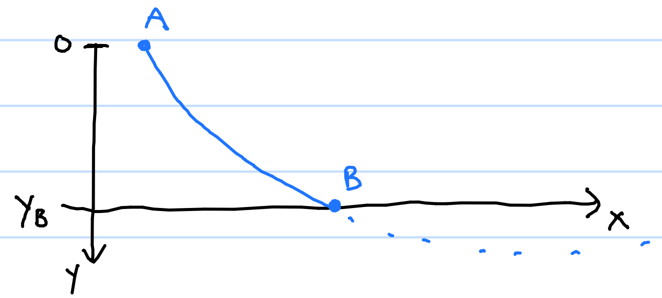
$$= \frac{1}{2c}\int_0^{\cos^{-1}(1-2cy)} d\theta\,\sin\theta\tan\left(\frac{\theta}{2}\right) = \frac{1}{2c}\int_0^{\cos^{-1}(1-2cy)} d\theta\,(1-\cos\theta)$$

trig identity

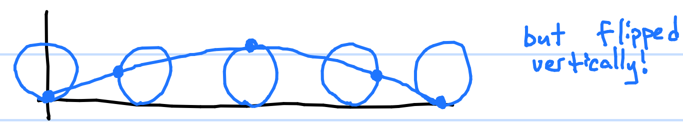
$$= \frac{1}{2c}\left[\theta - \sin\theta\right]_0^{\cos^{-1}(1-2cy)}$$

Thus, we have found x and y as parametric functions of θ:

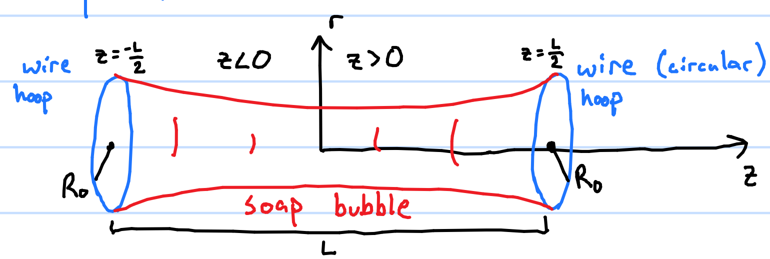
$$\boxed{\begin{aligned} x(\theta) &= \frac{1}{2c}\left[\theta - \sin\theta\right] \\ y(\theta) &= \frac{1}{2c}\left[1 - \cos\theta\right] \end{aligned}}$$



This curve is a cycloid, which is the path followed by a point on the outside of a wheel as it rolls:



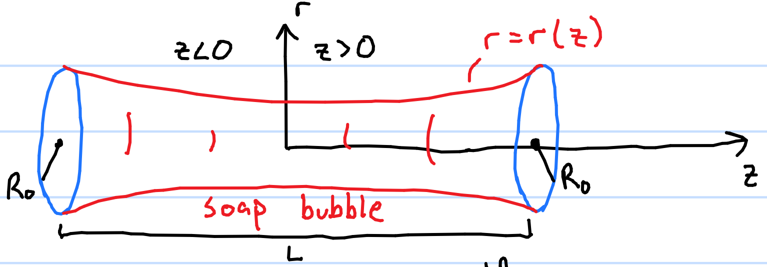
ex) Soap films



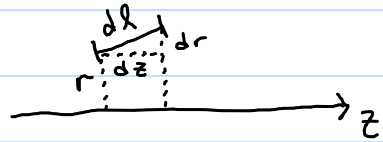
What is the shape of the soap bubble?

Soap bubble's energy dominated by surface tension

$\Rightarrow E_{tot} \sim \underbrace{\sigma}_{\text{surface tension}} A_{tot} \Rightarrow$ shape will have <u>minimal surface area</u>



S... surface area



$$dS = (2\pi r)\,d\ell$$

$$d\ell = \left[1+(r'(z))^2\right]^{1/2} dz$$

$\Rightarrow$ need to minimize $S = 2\pi \int_{-L/2}^{L/2} r\sqrt{1+r'^2}\,dz$

$\Rightarrow f(r, r', z) = r\sqrt{1+r'^2}$; E-L: $\dfrac{\partial f}{\partial r} - \dfrac{d}{dz}\left(\dfrac{\partial f}{\partial r'}\right) = 0$

$$\Rightarrow (1+r'^2)^{1/2} - \frac{d}{dz}\left(r r'(1+r'^2)^{-1/2}\right) = 0$$

(don't be afraid of Mathematica to check work!)

$$\Rightarrow \frac{1+(r')^2 - r r''}{(1+(r')^2)^{3/2}} = 0 \quad (*),$$

and since the denominator cannot be zero

$\Rightarrow \boxed{1+(r')^2 - r r'' = 0}$ second-order eqn to solve for $r(z)$

This could be done numerically. Alternatively, notice

$(*)$ is equivalent to

$$\frac{d}{dz}\left(\frac{r}{(1+r'^2)^{1/2}} = 0\right) \Rightarrow \frac{r}{(1+r'^2)^{1/2}} = c_1$$

$\Rightarrow \dfrac{dr}{dz} = \pm\sqrt{\dfrac{r^2}{c_1^2} - 1}$, first-order ODE easier to solve!

$\Rightarrow \displaystyle\int \frac{dr}{\pm\sqrt{\frac{r^2}{c_1^2}-1}} = \int dz$ check the inside cover of Taylor for a list of integrals

$u = \dfrac{r}{c_1} \Rightarrow du = \dfrac{dr}{c_1}$

$$\Rightarrow \pm c_1 \underbrace{\int \frac{du}{\sqrt{u^2-1}}}_{=\cosh^{-1}(u),\ \text{from inside cover}} = z + c_2$$

$\Rightarrow r(z) = c_1 \cosh\left[\dfrac{z+c_2}{c_1}\right]$ cosh is even $\Rightarrow \pm$ doesn't matter

Symmetry $r(z) = r(-z) \Rightarrow c_2 = 0$

$r(L/2) = r(-L/2) = c_1 \cosh\left[\frac{L}{2c_1}\right] = R_0$

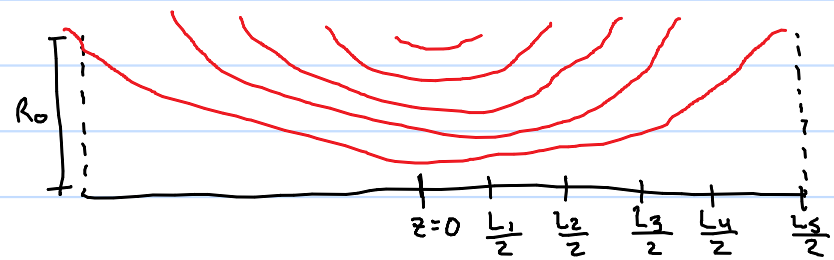
$\Rightarrow$ trancendental eqn to solve for c_1

$\Rightarrow$ numerically solve for $c_1 = c_1(L, R_0)$

$\Rightarrow \boxed{r(z) = c_1 \cosh\left[z/c_1\right]}$ These are called catenaries (suspension bridges, tow cables, etc.)

Numerics are carried out in Mathematica file Variational_Calculus.nb, courtesy of Mark Lusk, Colorado School of Mines.

We find, for fixed R_0 and varying L, caternaries look like:



At a certain value of L/R_0 (> 1.33), no soln for c_1 can be found! $\Rightarrow$ Only solns are soap films on each hoop.

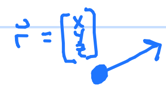
Extension of E-L to N dependent variables q_n and 1 independent variable t.

So far we have worked with

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \quad f = f(y, y', x)$$

In Lagrangian mechanics the state or "configuration" of a system is denoted $q_1, q_2, \ldots, q_n$, where n is the # degrees of freedom in the system.

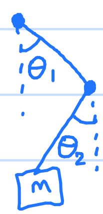
e.g.:

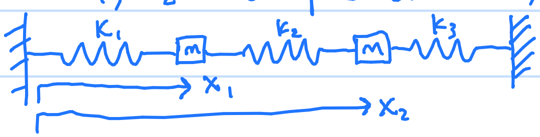


$q_1, q_2, q_3 = x, y, z$ (free particle)

$q_1, q_2 = \theta_1, \theta_2$ (double pendulum)

$q_1, q_2 = x_1, x_2$ (coupled oscillator)





Due to the number of guises these vars q_i can take, they are called <u>generalized coordinates</u>. The available values of the q_i define an n-dimensional <u>configuration space</u>. The time evolution of a system is a <u>trajectory</u> in this configuration space.

What if $f = f(q_1, q_2, \ldots, q_n, \dot q_1, \dot q_2, \ldots, \dot q_n, t)$?

Then you have E-L eqn for <u>each</u> q_i:

you prove this in HW1, #6,26

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot q_i}\right) = 0}$$

Multi-dim E-L eqns

(derivation assumes q_i may be varied independently)

These are N <u>coupled</u> 2^{nd} order ODE's.

OK. Ready for physics? Introduce the Lagrangian $\mathcal{L}$, and replace f by $\mathcal{L}$. Then call S the action and define it as

$$S := \int_{t_1}^{t_2} dt\, \mathcal{L}(q_1, \ldots, q_n, \dot q_1, \ldots, \dot q_n, t)$$

Look for stationary values of S. This is Lagrangian mechanics.

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right)}$$

Lagrange's Eqns

We will see $\mathcal{L} = (\text{kinetic energy}) - (\text{potential energy})$.

END OF CHAPTER 6

Chapter 7: Lagrangian Mechanics

Setting: Systems w/ conservative* forces

Recall (sections 4.2 - 4.5) that a force is conservative if:

① $\vec{F} = \vec{F}(\vec{r})$, and

② $\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{F}$ is independent of path

In this case, we can construct a potential energy function $U(\vec{r})$ such that $\vec{F} = -\nabla U$.

Also recall $\vec{F}$ conservative $\Leftrightarrow \nabla \times \vec{F} = 0$

ex) $\vec{F} = x\hat{x} + y\hat{y}$; $\nabla \times \vec{F} = 0$; $U = \frac{1}{2}(x^2 + y^2)$

ex) $\vec{F} = y\hat{x} + x\hat{y}$; $\nabla \times \vec{F} = \hat{z}(1-1) = 0$; $U = xy$

ex) $\vec{F} = x\hat{x} + x\hat{y}$; $\nabla \times \vec{F} = \hat{z}(1) \neq 0 \Rightarrow$ no U exists

Basically: friction (dissipation as heat, sound, etc) = nonconservative
everything else = conservative

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* not strictly required to be conservative, so long as there is a potential energy function. So $\vec{F} = \vec{F}(\vec{r}, t)$ w/ $\vec{F}(\vec{r}, t) = -\nabla U(\vec{r}, t)$ is fine, even though conservative forces strictly require $\vec{F} = \vec{F}(\vec{r})$ only, not $\vec{F}(\vec{r}, t)$. See (7.50) - (7.52), and section 4.5

Consider 1 particle w/ $U(x,y,z)$.
$\uparrow$ potential energy

Recall $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$
(T: kinetic energy)

Define **Lagrangian**, $\mathcal{L} := T - U$

$\Rightarrow \mathcal{L} = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)$

Define **Action**, $S := \int_{t_1}^{t_2} dt\, \mathcal{L}$

Hamilton's Principle:

The actual path followed by the particle between two points $\vec{r}_1$ and $\vec{r}_2$ over time interval $t_1 \to t_2$ is such that S is stationary.
↳ min, saddle, or max

Hamilton's principle is a special case of something more general: the **principle of least action**.

Next we will show that Hamilton's principle is equivalent to $\vec{F} = m\vec{a}$.

Let's explicitly evaluate Hamilton's principle:

If stationary $S = \int_{t_1}^{t_2} dt\, \mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z},t)$, then

E-L $\Rightarrow$ $\dfrac{\partial \mathcal{L}}{\partial x} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}}\right)$, etc. (eqns for x, y, z)

But $\mathcal{L} = T - U = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(x,y,z)$

$\Rightarrow -\dfrac{dU}{dx} = m\ddot{x}$, so w/ $F_x = -\frac{\partial U}{\partial x}$

$\Rightarrow \boxed{F_x = m\ddot{x}}$, and similarly for y, z

Thus, Hamilton's principle recovers Newtonian mechanics

The big advantage is that this works for any set of generalized coordinate $q_1, \dots, q_n$:

$$\underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\substack{\text{generalized}\\ \text{force}\\ F_i}} = \frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\substack{\text{generalized}\\ \text{momentum}\\ p_i}}\Big)$$

These may not even have units of force or momentum!

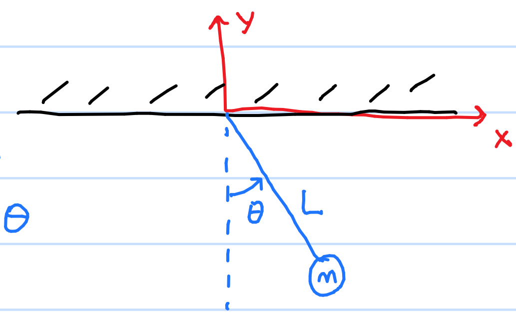
$\Rightarrow$ $\boxed{F_i = \dot{p}_i}$ Lagrange eqns in generalized coords

(sometimes the generalized force is written Q_i)

Choosing the "right" coords allows us to automatically take constraint forces into account.

Note: if $\dfrac{\partial \mathcal{L}}{\partial q_i} = 0$, we say momentum p_i is conserved.

Let's use this approach to study the simple pendulum.



$x = L \sin\theta$

$y = -L\cos\theta$

First we find $\mathcal{L} = \mathcal{L}(\theta, \dot{\theta}, t) = T - U$

$T = \frac{m}{2}(\dot{x}^2+\dot{y}^2) = \frac{m}{2}L^2\left((\cos\theta\,\dot{\theta})^2 + (\sin\theta\,\dot{\theta})^2\right)$

$= \frac{1}{2}mL^2\dot{\theta}^2$

$U = mgy = -mgL\cos\theta$

$\Rightarrow \mathcal{L}(\theta,\dot{\theta},t) = \frac{1}{2}mL^2\dot{\theta}^2 + mgL\cos\theta$

$\Rightarrow \mathcal{L}(\theta,\dot\theta,t) = \frac{1}{2} mL^2\dot\theta^2 + mgL\cos\theta$, and

E-L: $\dfrac{\partial \mathcal{L}}{\partial \theta} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot\theta}\right)$

$$\Rightarrow \underbrace{-mgL\sin\theta}_{\substack{\text{generalized}\\ \text{force}\\ \text{(torque!)}}} = \frac{d}{dt}\underbrace{\left(mL^2\dot\theta\right)}_{\substack{\text{generalized}\\ \text{momentum}\\ \text{(angular momentum!)}}}$$

$\Rightarrow \ddot\theta = \dfrac{-g}{L}\sin\theta$ Eqn of motion

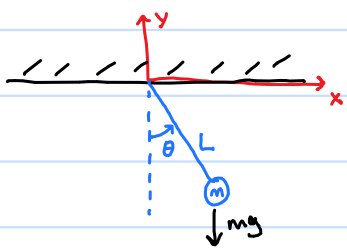
Small angle approx: $\theta \ll 1 \Rightarrow \sin\theta \approx \theta$

$\Rightarrow \ddot\theta = \frac{-g}{L}\theta$

$\Rightarrow \theta(t) = \theta_0\cos(\omega t)$, $\omega = \sqrt{g/L}$, $\theta(0) = \theta_0$

assuming $\dot\theta(0) = 0$

How would we have solved this before?



$$\underbrace{\vec\tau}_{\text{torque}} = \underbrace{\frac{d\vec L}{dt}}_{\Delta \text{ ang. mom.}}$$

$\vec\tau = \vec r \times \vec F = -L\,mg\,\sin\theta\ \hat z$

$\vec L = \underbrace{I}_{\substack{\text{moment}\\ \text{of inertia}}}\vec\omega = \underbrace{(mL^2)}_{\text{MoI for point particle}}(\dot\theta)\ \hat z$

$$\Rightarrow \boxed{-Lmg\sin\theta = \frac{d}{dt}\left(mL^2\dot\theta\right)}$$

EXACT SAME

From this example, we see that the generation of angular momentum by torque,

$$\vec\tau = \frac{d\vec L}{dt},$$

is simply one realization of Lagrange's equations for the choice of coordinate $q = \theta$.

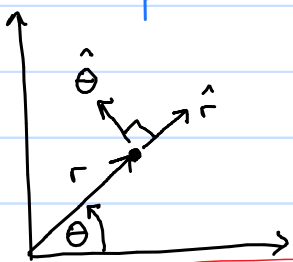
Let's try one more trivial example:

One particle in 2-dimensions w/ polar coords

$\hat r = \cos\theta\ \hat x + \sin\theta\ \hat y$

$\hat\theta = -\sin\theta\ \hat x + \cos\theta\ \hat y$

$\mathbf{r} = r\hat r = r\cos\theta\ \hat x + r\sin\theta\ \hat y$

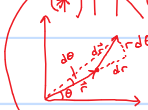


$\dot{\vec r} = (\dot r\cos\theta - r\sin\theta\,\dot\theta)\,\hat x$

$\quad + (\dot r\sin\theta + r\cos\theta\,\dot\theta)\,\hat y$

$\quad = \dot r\,\hat r + r\dot\theta\,\hat\theta$

(*) PROOF:



$\Rightarrow d\vec r = dr\,\hat r + r\,d\theta\,\hat\theta$

$df = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta$

$= \nabla f \cdot d\vec r$

$\Rightarrow \nabla f = \frac{\partial f}{\partial r}\hat r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat\theta$

$\Rightarrow T = \frac{m}{2}(\dot x^2 + \dot y^2) = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2)$

Assume $U = U(r,\theta) \Rightarrow \mathcal{L} = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - U$

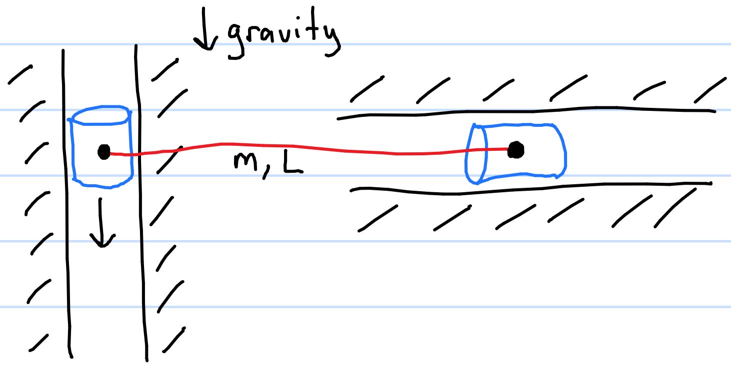
E-L $\Rightarrow \dfrac{\partial \mathcal{L}}{\partial r} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot r}\right)$ and $\dfrac{\partial \mathcal{L}}{\partial \theta} = \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot\theta}\right)$

$\Rightarrow mr\dot\theta^2 - \dfrac{\partial U}{\partial r} = m\ddot r$ and $-\dfrac{\partial U}{\partial \theta} = mr^2\ddot\theta + 2mr\dot r\dot\theta$

In polar (*) $\nabla U = \frac{\partial U}{\partial r}\hat r + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat\theta \Rightarrow$ Let $F_r = \frac{-\partial U}{\partial r}$, $F_\theta = \frac{-1}{r}\frac{\partial U}{\partial \theta}$

$$\Rightarrow F_r = \underbrace{m\ddot r}_{\text{radial accel}} - \underbrace{mr\dot\theta^2}_{\text{centripetal accel}}\ ;\quad F_\theta = \underbrace{mr\ddot\theta}_{\text{tangential accel}} + \underbrace{2m\dot r\dot\theta}_{\text{Coriolis accel}}$$

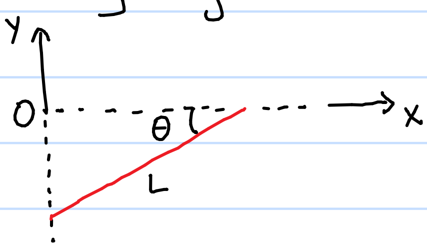
ex) 1 degree of freedom system



A thin rod (in red) starts horizontally at rest and moves under the influence of gravity. The blue pistons have negligible and are frictionless. How long does the rod take to become vertical?

(We will use this problem to generate step-by-step methodology for solving Lagrangian mechanics problems)

Step 1: Identify generalized coordinates



Choose $q = \theta$.

Step 2: Calculate $T, U, \mathcal{L} = T - U$

$$T = \frac{1}{2} m v_{com}^2 + \frac{1}{2} I_{com} \omega^2$$

$\omega = \dot{\theta}$

moment of inertia for thin rod: $\frac{1}{12} m L^2$

$$\vec{r}_{com} = \frac{L}{2}\left[\cos\theta\,\hat{x} - \sin\theta\,\hat{y}\right]$$

$$\Rightarrow \dot{\vec{r}}_{com} = \frac{L\dot{\theta}}{2}\left[-\sin\theta\,\hat{x} - \cos\theta\,\hat{y}\right]$$

$$\Rightarrow v_{com}^2 = \left(\frac{L^2}{4}\right)\dot{\theta}^2$$

$$\Rightarrow T = mL^2\dot{\theta}^2\left[\frac{1}{8} + \frac{1}{24}\right] = mL^2\dot{\theta}^2/6$$

$$U = mg\,y_{com} = -mgL\sin\theta/2$$

$$\Rightarrow \mathcal{L} = mL^2\dot{\theta}^2/6 + mgL\sin\theta/2$$

Step 3: Calculate generalized forces + momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow p_i = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{mL^2\dot{\theta}}{3}$$

units of momentum?

$kg \cdot m^2/s \rightarrow$ no

$$Q_i = \frac{\partial \mathcal{L}}{\partial q_i} \Rightarrow Q_i = \frac{\partial \mathcal{L}}{\partial \theta} = \frac{mgL\cos\theta}{2}$$

generalized force

units of force? $kg \cdot m^2/s^2 \rightarrow$ no

Step 4: Construct Lagrange Eqns

$$Q_i = \dot{p}_i$$

$$\Rightarrow \frac{mgL\cos\theta}{2} = \frac{d}{dt}\left(\frac{mL^2\dot{\theta}}{3}\right) = \frac{mL^2\ddot{\theta}}{3}$$

$$\Rightarrow \ddot{\theta} = \frac{3g}{2L}\cos\theta$$

Step 5: Non-dimensionalize Lagrange eqns

$$t = t_c\tau \Rightarrow \ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{1}{t_c^2}\frac{d^2\theta}{d\tau^2} = \frac{3g}{2L}\cos\theta$$

Choose $t_c = \sqrt{2L/3g}$ (units $= \sqrt{m/m/s^2} = s$ ✓)

$$\Rightarrow \frac{d^2\theta}{d\tau} = \cos\theta. \text{ Recycle } \tau \to t$$

$$\Rightarrow \ddot{\theta} = \cos\theta$$

Step 6: Solve ND Lagrange eqns

$$\ddot{\theta} = \cos\theta, \quad \theta(0) = 0, \quad \dot{\theta}(0) = 0$$

$$\Rightarrow \dot{\theta}\ddot{\theta} = \cos\theta\,\dot{\theta} \quad \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2\right) = \frac{d}{dt}(\sin\theta)$$

$$\Rightarrow \dot{\theta}^2 = \sin\theta + c_1. \text{ @ } t=0,\ \dot{\theta} = \theta = 0 \Rightarrow c_1 = 0$$

$$\Rightarrow \dot{\theta} = \sqrt{2\sin\theta}$$

$$\Rightarrow \int_0^{\pi/2} \frac{d\theta}{\sqrt{2\sin\theta}} = \int_{\text{horizontal}}^{\text{vertical}} dt = t_{final}$$

Must solve numerically! $\Rightarrow t_{final} = 1.854$

Step 7: Redimensionalize

$$t = t_c\tau \Rightarrow \boxed{t_{final} = 1.854\sqrt{2L/3g}}$$

(check! units work)

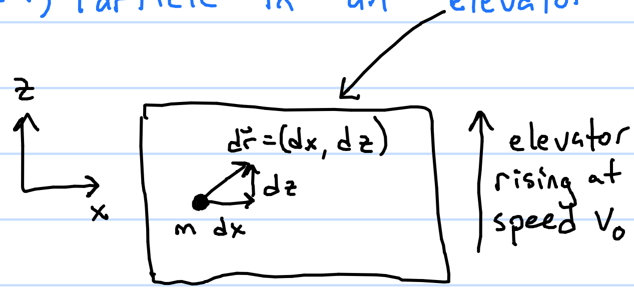
Having established that Lagrange's equations $\left(\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right)$ recover Newton's laws, let us now try to <u>motivate</u> why the Lagrangian $\mathcal{L} = T - U$ is so special.

To do this, we introduce the <u>virtual displacement</u> $\delta \vec{r}_i$, an infinitesimal change of coordinates that obeys constraints and is executed <u>instantaneously</u>. i.e. $\delta t = 0$
Note $\delta \vec{r}_i$ are distinguished from the actual motion of a particle $d\vec{r}$ occuring in the infinitesimal time dt.

$\delta \leftrightarrow$ virtual $\qquad d \leftrightarrow$ actual

(Mathematically, we treat both the same way
$\Rightarrow \delta$ obeys chain rule + product rule)

Ex) Particle in an elevator



$q = x$
$\delta t = 0$

$d\vec{r} = (dq, \; v_0\, dt)$
$\delta \vec{r} = (\delta q, \; 0)$

Define the <u>virtual work</u> $\delta W_i = \vec{F}_i \cdot \delta \vec{r}_i$, which says "force does work when a particle moves in its direction."

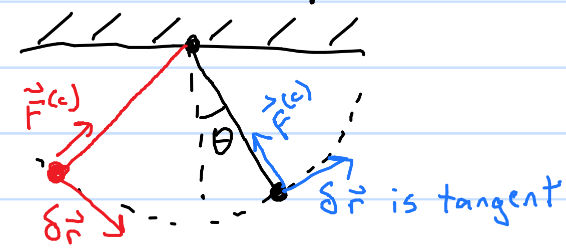
Decompose forces into <u>applied</u> and <u>constraint</u> forces:
$\vec{F}_i = \vec{F}_i^{(a)} + \vec{F}_i^{(c)}$, and note $\vec{F}_i = m\ddot{\vec{r}}_i$ implies
$\sum_i \vec{F}_i^{(a)} + \vec{F}_i^{(c)} - m\ddot{\vec{r}}_i = 0$

$$\Rightarrow \sum_i \left(\vec{F}_i^{(a)} - m\ddot{\vec{r}}_i\right)\cdot \delta \vec{r}_i + \sum_i \vec{F}_i^{(c)} \cdot \delta \vec{r}_i = 0 \qquad (*)$$

sum is over $i = 1, \ldots, N$ particles

Let's isolate and think about $\sum_i \vec{F}_i^{(c)} \cdot \delta \vec{r}_i$:

Ex) Plane pendulum



When we have virtual displacements $\delta \vec{r}$ that are consistent w/ constraints, we have
$\sum_i \vec{F}_i^{(c)} \cdot \delta \vec{r}_i = 0.$

This follows from the <u>principle of virtual work</u>. (next page) With $(*)$, this immediately leads to <u>d'Alembert's principle</u>:

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right)\cdot \delta \vec{r}_i = 0$$

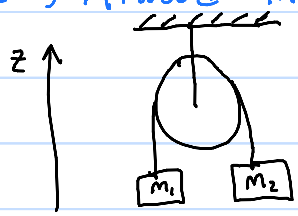
(Note how the constraint force is absent)

Before proceeding, let's flesh out the principle of virtual work. This principle states that at static equilibrium, the virtual work done by all applied forces is zero

$$\Leftrightarrow \quad \delta W = \sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0.$$

Newton's 3rd law states @ equilibrium, applied forces are equal and opposite to the constraint forces. Hence the virtual work of the constraint forces must be zero as well.

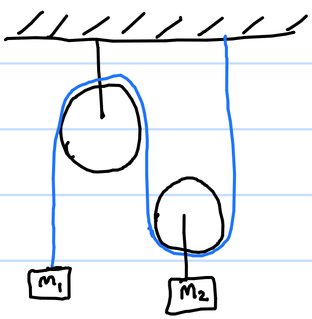
Ex) Atwood machines



At equilibrium:

$$m_1 g(\delta z) + m_2 g(-\delta z) = 0$$

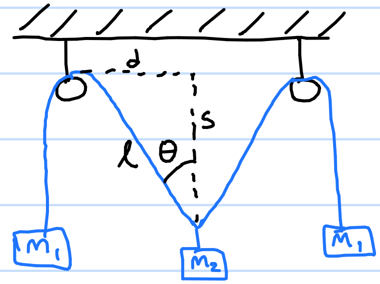
$$\Rightarrow \underline{m_1 = m_2}$$



at equilibrium:

$$m_1 g(\delta z) - m_2 g(\delta z/2) = 0$$

$$\Rightarrow m_2 = 2m_1$$



at equilibrium:

$$\ell = \frac{d}{\sin\theta}, \qquad s = \frac{d}{\tan\theta}$$

$$\delta s = \frac{-d}{\tan^2\theta} \frac{\delta\theta}{\cos^2\theta} = \frac{-d}{\sin^2\theta}\delta\theta$$

$$\delta \ell = \frac{-d}{\sin^2\theta}\cos\theta\, \delta\theta$$

$$\Rightarrow m_2 g\, \delta s + 2m_1 g(-|\delta \ell|) = 0$$

$$\Rightarrow \frac{-d\, m_2}{\sin^2\theta}\delta\theta + \frac{2 d\, m_1}{\sin^2\theta}\cos\theta\, \delta\theta = 0$$

$$\Rightarrow m_2 = 2m_1 \cos\theta \quad \Rightarrow \quad \cos\theta = \frac{m_2}{2m_1}$$

Thus, principle of virtual work gives an easy way to find equilibrium configurations.

Now let's return to d'Alembert's principle:

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right) \cdot \delta \vec{r}_i = 0 \qquad (\Delta)$$

But what if we want to work in generalized coordinates? Need to know transformation rules:

$$\vec{r}_i = \vec{r}_i(\underbrace{q_1, \ldots, q_s}_{\text{gen'l coords}}, t), \quad w/ \quad s = 3\underset{\uparrow}{N} - K^{\leftarrow \#\text{constraints}}$$

N particles w/ x, y, z

$$\dot{\vec{r}}_i = \sum_{j=1}^{s} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} = \dot{\vec{r}}_i(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t)$$

$$\Rightarrow \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$, which says the "dots cancel"

Since a virtual displacement occurs at $\delta t = 0$,

$$\delta\vec{r}_i = \sum_{j=1}^{s} \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

We may substitute this into (Δ):

$$\sum_{i=1}^{N} \vec{F}_i^{(a)}\cdot\delta\vec{r}_i = \sum_{i=1}^{N}\sum_{j=1}^{s} \vec{F}_i^{(a)}\cdot\frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

$$:= \sum_{j=1}^{s} Q_j\,\delta q_j,$$

where we define the generalized force

$$Q_j := \sum_{i=1}^{N} \vec{F}_i^{(a)}\cdot\frac{\partial \vec{r}_i}{\partial q_j}$$

"the component of the force in the $\hat{q}_j$ direction"

gradient for ith particle only

For conservative systems $\vec{F}_i^{(a)} = -\nabla_i U(\vec{r}_1,\dots,\vec{r}_N)$

$$\Rightarrow Q_j = \sum_{i=1}^{N} -(\nabla_i U)\cdot\frac{\partial \vec{r}_i}{\partial q_j}$$

$$= \sum_{i=1}^{N} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{\partial U}{\partial r_{ix}} \\ \frac{\partial U}{\partial r_{iy}} \\ \frac{\partial U}{\partial r_{iz}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \frac{\partial \vec{r}_i}{\partial q_j} \overset{\text{chain rule}}{=} \frac{-\partial U}{\partial q_j}$$

$$\Rightarrow \sum_{i=1}^{N} \vec{F}_i^{(a)}\cdot\delta\vec{r}_i = \sum_{j=1}^{s} -\frac{\partial U}{\partial q_j}\,\delta q_j \qquad (\#1)$$

$\therefore$ converted this term into gen'l coords.

Now let's focus on the second term of (Δ).

First, notice

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \sum_{\ell=1}^{s} \frac{\partial^2 \vec{r}_i}{\partial q_j\,\partial q_\ell}\,\dot{q}_\ell + \frac{\partial^2 \vec{r}_i}{\partial q_j\,\partial t}$$

$$= \frac{\partial}{\partial q_j}\left(\sum_{\ell=1}^{s} \frac{\partial \vec{r}_i}{\partial q_\ell}\,\dot{q}_\ell + \frac{\partial \vec{r}_i}{\partial t}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

Then, consider the term in (Δ)

$$\sum_{i=1}^{N} \dot{\vec{p}}_i\cdot\delta\vec{r}_i = \sum_{i=1}^{N} m_i\,\ddot{\vec{r}}_i\cdot\delta\vec{r}_i = \overbrace{\sum_{i=1}^{N}\sum_{j=1}^{s} m_i\,\ddot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j}^{\equiv(*)}$$

Use $\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right) = \ddot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$

$$(*) = \sum_{i=1}^{N}\sum_{j=1}^{s} m_i\left[\frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right]\delta q_j$$

Use $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{\vec{r}}_i^{\,2}\right)\right) = \frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) \overset{\text{from prev page}}{=} \frac{d}{dt}\left(\dot{\vec{r}}_i\cdot\frac{\partial \vec{r}_i}{\partial q_j}\right)$,

and similarly $\frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\vec{r}}_i^{\,2}\right) = \dot{\vec{r}}_i\cdot\frac{\partial \dot{\vec{r}}_i}{\partial q_j}$

$$(*) = \sum_{i=1}^{N}\sum_{j=1}^{s} m_i\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{\vec{r}}_i^{\,2}\right)\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\vec{r}}_i^{\,2}\right)\right]\delta q_j$$

But: $\sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = T$, the kinetic energy

$$\Rightarrow \sum_{i=1}^{N} \dot{\vec{p}}_i\cdot\delta\vec{r}_i = \sum_{j=1}^{s}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j \qquad (\#2)$$

Now, combine $(\#1)$ and $(\#2)$ w/ (Δ) from prev page:

$$(\Delta) = \sum_{i=1}^{N}\left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0$$

$$\Leftrightarrow \sum_{j=1}^{s}\left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} - \frac{\partial U}{\partial q_j}\right]\delta q_j = 0$$

Finally, a conservative potential $U = U(q_1,\dots,q_s) \Rightarrow \frac{\partial U}{\partial \dot{q}_j} = 0$.

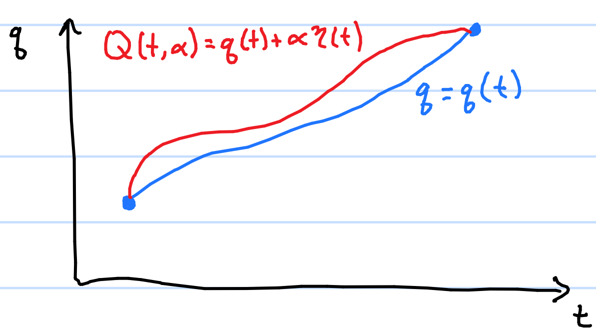
Thus

$$\boxed{\sum_{j=1}^{s}\left[\frac{\partial (T-U)}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j}\right)\right]\delta q_j = 0}$$

Therefore, we see the Lagrangian $\mathcal{L} = T - U$ naturally arises from d'Alembert's principle + the principle of virtual work.

Lagrangian mechanics conveniently allows us to ignore constraint forces by choosing an appropriate set of generalized coordinates. But what if we need to know these forces? E.g. I wish to know tension on a pendulum to ensure the string doesn't break, or I wish to ensure my roller coaster meets federal regulations.

It is convenient to do this using variational notation. Recall:



$q(t)$... trial function for stationary action S

$$\frac{dS}{d\alpha} = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial q} \underbrace{\frac{\partial Q}{\partial \alpha}}_{\eta} + \frac{\partial \mathcal{L}}{\partial \dot{q}} \underbrace{\frac{\partial \dot{Q}}{\partial \alpha}}_{\dot{\eta}} \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right) \frac{\partial Q}{\partial \alpha}$$

Let $\delta S := \frac{dS}{d\alpha}\delta\alpha$ Variation in S

$\delta q := \frac{\partial Q}{\partial \alpha}\delta\alpha$ Variation in q

We can now write:

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \right) \delta q$$

When there are multiple q's,

$$\delta S = \int_{t_1}^{t_2} dt \underbrace{\sum_{i=1}^{N} \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right) \delta q_i}_{} = 0$$

If the q_i's are independent, then each of these terms must be zero. Otherwise we must relate them before going further.

Before applying constraints to Lagrangian mechanics, let's briefly recall how constraints work in ordinary calculus.

Ex) Minimize $z = x^2 + y^2$ as fcn of x and y

$\frac{\partial z}{\partial x} = 2x = 0$ and $\frac{\partial z}{\partial y} = 2y = 0 \Rightarrow (x,y) = (0,0)$

(can check second derivatives all ≥ 0) (#1)

Ex) Minimize $z = x^2 + y^2$ subject to constraint $2x + y = 1$

$y = 1 - 2x \Rightarrow z = x^2 + (1-2x)^2 = 5x^2 - 4x + 1$

$\frac{\partial z}{\partial x} = 10x - 4 = 0 \Rightarrow x = \frac{2}{5}, \; y = \frac{1}{5}$ (#2)

How could we have solved this w/ Lagrange multipliers?

Apply to generic function $f = f(y_1, \ldots, y_N) = f(y_j)$.

With no constraints, $\delta f = \sum_{j=1}^{N} \frac{\partial f}{\partial y_j} \delta y_j = 0$ at equilibrium; since variation δy_j is arbitrary $\Rightarrow \frac{\partial f}{\partial y_j} = 0 \;\; \forall \, y_j$

compare to #1

With constraint $\phi(y_1, \ldots, y_N) = 0$, δy_j are related to each other. 1 constraint $\Rightarrow$ only $N-1$ coords independent.

$$\delta f = \sum_{j=1}^{N} \frac{\partial f}{\partial y_j} \delta y_j = 0 \; ; \quad \delta \phi = \sum_{j=1}^{N} \frac{\partial \phi}{\partial y_j} \delta y_j = 0$$

Idea: Use elimination to remove δy_N term.

Multiple ϕ eqn by $\lambda := \frac{-\partial f / \partial y_N}{\partial \phi / \partial y_N}$ and add to f eqn:

$$\delta f = \sum_{j=1}^{N} \left[\frac{\partial f}{\partial y_j} + \lambda \frac{\partial \phi}{\partial y_j} \right] \delta y_j$$

note $j=N$ returns $\frac{\partial f}{\partial y_N} - \frac{\partial f / \partial y_N}{\partial \phi / \partial y_N} \frac{\partial \phi}{\partial y_N} = 0$

$$\Rightarrow \delta f = \sum_{j=1}^{N-1} \left[\frac{\partial f}{\partial y_j} + \lambda \frac{\partial \phi}{\partial y_j} \right] \delta y_j = 0$$

Now $y_1, \ldots, y_{N-1}$ are all independent

$$\Rightarrow \frac{\partial f}{\partial y_j} + \lambda \frac{\partial \phi}{\partial y_j} = 0 \quad \text{for } j = 1, \ldots, N-1$$

Ex) Minimize $z = x^2 + y^2$ subject to constraint $2x + y = 1$

$\phi(x,y) = 2x + y - 1$. Choose $y_N = y \Rightarrow \lambda = \frac{-\partial z / \partial y}{\partial \phi / \partial y} = \frac{-2y}{1} = -2y$

$$\Rightarrow \frac{\partial z}{\partial x} + (-2y) \frac{\partial \phi}{\partial x} = 2x + (-2y)(2) = 2x - 4(1-2x) = 0$$

$\Rightarrow 10x = 4 \Rightarrow x = \frac{2}{5}, \; y = \frac{1}{5}$

ALTERNATIVELY: Assume we don't know λ

$\Rightarrow \frac{\partial}{\partial x}[z + \lambda \phi] = \frac{\partial}{\partial x}\left[x^2 + y^2 + \lambda(2x + y - 1)\right] = 2x + 2\lambda = 0$

$\Rightarrow x = -\lambda$

$\frac{\partial}{\partial y}\left[x^2 + y^2 + \lambda(2x + y - 1)\right] = 2y + \lambda = 0.$

Constraint $\Rightarrow -2\lambda - \frac{\lambda}{2} - 1 = 0 \Rightarrow \lambda = \frac{-2}{5} \Rightarrow x = \frac{2}{5}, \; y = \frac{1}{5}$

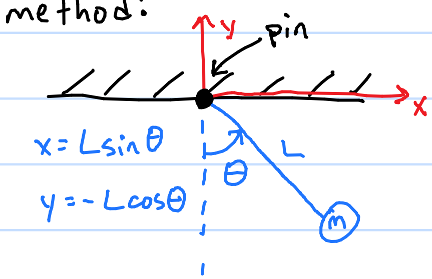
This may be generalized if there are K constraints:

$$\delta f = \sum_{i=1}^{N-K} \left[\frac{\partial f}{\partial y_i} + \sum_{j=1}^{K} \lambda_j \frac{\partial \phi_j}{\partial y_i} \right] \delta y_i ,$$

w/ constraints characterized by ϕ_K, and λ_K are solved for

In this previous approach the Lagrange multiplier allows us to cleverly "add zero" in order to encode a constraint. We pursue a similar course here.

We will use the plane pendulum to explain this method:



Standard approach:

$$\mathcal{L} = \frac{1}{2} m L^2 \dot{\theta}^2 + mgL\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)$$

$$\Rightarrow -mgL\sin\theta = mL^2\ddot{\theta} \Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$$

Now we will redo this problem using coords x and y. Goal: find force exerted on pin ($F_{centripetal}$).

Now x and y are related by a constraint equation: $W(x,y) = L^2 - (x^2+y^2) = 0$

- What are E-L equations now?
- What do we gain by including constraint eqns? constraint forces

Look for stationary action, S:

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}(x, y, \dot{x}, \dot{y}, t) \quad w/ \quad \delta S = 0$$

$$\Rightarrow \int_{t_1}^{t_2} dt \sum_{i=1}^{2} \left[\left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right)\delta q_i\right] = 0, \quad (*)$$

with $q_1 = x$ and $q_2 = y$.

Normally at this point we say "δq_i are arbitrary so we need $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$ for $i = 1, 2$."
This is not true in our current scenario, since x and y and therefore δx and δy are related by the constraint eqn.

Thus, we need

$$\delta W = \frac{\partial W}{\partial x}\delta x + \frac{\partial W}{\partial y}\delta y = 0.$$

requires variations δx and δy to be compatible w/ constraints

Note $\lambda(t)\,\delta W = 0$, too. Add this into $(*)$:

Lagrange multiplier

$$\Rightarrow \int_{t_1}^{t_2} dt\left(\frac{\partial \mathcal{L}}{\partial x} + \lambda(t)\frac{\partial W}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\right)\delta x$$

$$+ \int_{t_1}^{t_2} dt\left(\frac{\partial \mathcal{L}}{\partial y} + \lambda(t)\frac{\partial W}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\right)\delta y = 0$$

Now, the trick of this method is to choose $\lambda(t)$ so that

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda(t)\frac{\partial W}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0. \quad (1)$$

Note we are choosing $\lambda(t)$ to make this happen, not enforcing $\delta S = 0$ like we usually do. Then, only after this term is zero, do we enforce $\delta S = 0$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial y} + \lambda(t)\frac{\partial W}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = 0. \quad (2)$$

Also, $W(x,y) = L^2 - (x^2+y^2) = 0. \quad (3)$

Thus, we have 3 equations (1), (2), (3) and 3 unknowns $x(t)$, $y(t)$, and $\lambda(t)$.

Thus, we may solve for $x(t)$, $y(t)$, and $\lambda(t)$.

$\lambda(t)$... Lagrange multiplier that allows us to derive force balances for x and y even though they are not independent. It add constraint forces to the balances.

Let's interpret the $\lambda(t)\frac{\partial W}{\partial x}$ and $\lambda(t)\frac{\partial W}{\partial y}$ terms more:

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda(t)\frac{\partial W}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right).$$

U was defined as the potential energy for non-constraint forces

If we use a one-dimensional particle to guide our intuition, w/ $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$, and $F_x = \frac{-\partial U}{\partial x}$,

$$\Rightarrow \underbrace{F_x}_{\text{applied force}} + \underbrace{\lambda(t)\frac{\partial W}{\partial x}}_{???} = m\ddot{x}$$

$$\vec{F}_{tot,x} = \vec{F}_{applied,x} + \vec{F}_{constraint,x} = m\ddot{x}$$

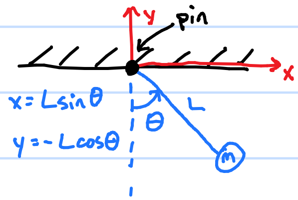
Note the force derived from the potential is an applied force. Thus subtracting $F_{applied}$ from both sides yields:

$$\boxed{\lambda(t)\frac{\partial W}{\partial x} = F_{constraint,x}}$$

That is, by solving for the Lagrange multiplier $\lambda(t)$ we will immediately solve for $F_{constraint}$ as well.

$\mathcal{L} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy$, $W = L^2 - (x^2+y^2)$

Let us return now to the plane pendulum.



(1) $0 + \lambda(t)(-2x) - \frac{d}{dt}(m\dot{x}) = 0$

(2) $-mg + \lambda(t)(-2y) - \frac{d}{dt}(m\dot{y}) = 0$

(3) $L^2 - (x^2+y^2) = 0$

$$\Rightarrow m\ddot{x} = -2\lambda x$$
$$m\ddot{y} = -mg - 2\lambda y$$
$$x^2+y^2 = L$$

constraint forces

To evaluate the constraint forces we must compute λ.

$$\Rightarrow m\ddot{x}x + m\ddot{y}y = -2\lambda x^2 - mgy - 2\lambda y^2$$
$$= -mgy - 2\lambda(\underbrace{x^2+y^2}_{L^2})$$

$$\Rightarrow \lambda = \frac{-m}{2L^2}(x\ddot{x} + y\ddot{y} + gy)$$

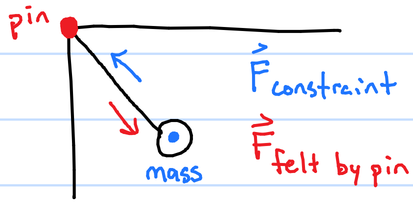
Note if $x^2+y^2 = L^2$ then

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(L^2) \Rightarrow \cancel{2}x\dot{x} + \cancel{2}y\dot{y} = 0$$
$$\Rightarrow x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0$$

$$\Rightarrow \lambda = \frac{-m}{2L^2}(-\dot{x}^2 - \dot{y}^2 + gy)$$

We can solve for $\dot{x}$ and $\dot{y}$ easily.

Our goal has been to find the force exerted on the pin:



$$\vec{F}_{pin} = -\vec{F}_{constraint}$$

$$\Rightarrow |\vec{F}_{pin}| = -\vec{F}_{constraint} \cdot \underbrace{\hat{r}}_{\frac{x}{L}\hat{x} + \frac{y}{L}\hat{y}}$$

$$= -\begin{bmatrix} -2\lambda x \\ -2\lambda y \end{bmatrix} \cdot \begin{bmatrix} x/L \\ y/L \end{bmatrix} = \frac{2\lambda}{L}\left(x^2 + y^2\right) = 2\lambda L$$

$$= \frac{m}{L}\left(\dot{x}^2 + \dot{y}^2 - gy\right).$$

use $\lambda = \frac{-m}{2L^2}\left(-\dot{x}^2 - \dot{y}^2 + gy\right)$

$$= \frac{m}{L}\left(L^2\dot{\theta}^2 + gL\cos\theta\right)$$

$x = L\sin\theta \quad y = L\cos\theta$

$\dot{x} = L\cos\theta\,\dot{\theta}$

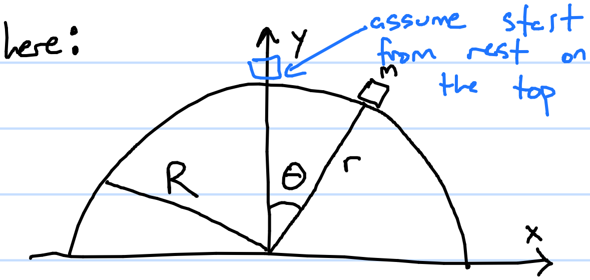
$\dot{y} = L\sin\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = L^2\dot{\theta}^2$

$$\boxed{\Rightarrow |\vec{F}_{pin}| = \underbrace{mL\dot{\theta}^2}_{\text{centripetal force}} + \overbrace{mg\cos\theta}^{\text{gravity}}}$$

$\therefore$ Force exerted on a pin is combination of centripetal and gravitational force (We could solve for $\theta(t)$ explicitly if we wanted using standard Lagrange equations)

Next, let's apply our new method to find the angle that an ice cube falls off a hemisphere:



Choose gen'l coords r, θ

Constraint eqn: $W(r,\theta) = r - R = 0$

↳ constraint forces related to constraint eqn $\Rightarrow$ when constraint force $= 0$, ice cube falls off

$x = r\sin\theta \quad y = r\cos\theta$

$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

$\dot{x} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$\dot{y} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

Constraint eq $\Rightarrow$ $\underbrace{\dot{r} = \ddot{r} = 0}_{\text{until ice cube leaves hemisphere}}$

$\Rightarrow T = \frac{1}{2}m\left(r^2\dot{\theta}^2\right)$

$U = mgy = mgr\cos\theta$

$\mathcal{L} = \frac{1}{2}mr^2\dot{\theta}^2 - mgr\cos\theta$

$W = r - R = 0$

$\mathcal{L} = \frac{1}{2} m r^2 \dot\theta^2 - mgr\cos\theta \qquad W = r - R = 0$

Plug these into the constrained E-L equations

$$\frac{\partial \mathcal{L}}{\partial \theta} + \lambda \frac{\partial W}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = mgr\sin\theta - mr^2\ddot\theta = 0$$

$$\frac{\partial \mathcal{L}}{\partial r} + \lambda \frac{\partial W}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) = mr\dot\theta^2 - mg\cos\theta + \lambda(1) = 0$$

constraint force

$$\Rightarrow \lambda = mg\cos\theta - mr\dot\theta^2 ; \qquad \ddot\theta = \frac{g}{r}\sin\theta = \frac{g}{R}\sin\theta$$

before falling off

Want to find when $\lambda = 0$ ⇒ need to solve for $\dot\theta$. Multiply $\ddot\theta$ eqn by $\dot\theta$:

$$\dot\theta\ddot\theta = \frac{g}{R}\sin\theta\,\dot\theta \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot\theta^2\right) = \frac{d}{dt}\left(\frac{-g}{R}\cos\theta\right)$$

$$\Rightarrow \frac{\dot\theta^2}{2} = \frac{-g}{R}\cos\theta + c.$$

Use $\dot\theta(0) = \theta(0) = 0$

$\Rightarrow c = g/R$

$$\Rightarrow \dot\theta^2 = \frac{2g}{R}(1-\cos\theta)$$

Plug this into λ to find

$$\lambda = mg\cos\theta - \frac{m\cancel{r}\,2g}{\cancel{R}}(1-\cos\theta)$$

$$= mg(3\cos\theta - 2)$$

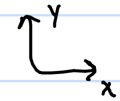
note when $\theta = 0$ (at top) constraint force is just mg

Ice cube falls off when $\lambda = 0$

$$\Rightarrow \theta_0 = \cos^{-1}(2/3) \approx 48.2^\circ$$

(slightly more than halfway down)

Ex) Disk rolling on an inclined plane



Choose gen'l coords y and θ.

$y = z\sin\alpha$

$x = z\cos\alpha$

Constraint eq? Yes: $W(y,\theta) = z - R\theta = 0$

Kinetic energy $T = \frac{1}{2}m\dot z^2 + \frac{1}{2}I\dot\theta^2$ $\quad I_{disk} = \frac{1}{2}mR^2$

$$= \frac{1}{2}m\dot z^2 + \frac{1}{4}mR^2\dot\theta^2$$

Potential energy $U = -mgz\sin\alpha$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\dot z^2 + \frac{1}{4}mR^2\dot\theta^2 + mgz\sin\alpha$$

Alternative/equivalent method for constrained E-L eqns:

Define $\tilde{\mathcal{L}} = \mathcal{L} + \lambda W$, and use standard E-L.

$$\tilde{\mathcal{L}} = \frac{1}{2}m\dot z^2 + \frac{1}{4}mR^2\dot\theta^2 + mgz\sin\alpha + \lambda(z - R\theta)$$

$$\Rightarrow \frac{\partial \tilde{\mathcal{L}}}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot\theta}\right) = 0 \Rightarrow -\cancel{R}\lambda - \frac{1}{2}mR^{\cancel{2}}\ddot\theta = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial z} - \frac{d}{dt}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\dot z}\right) = 0 \Rightarrow mg\sin\alpha + \lambda - m\ddot z = 0$$

Add eqs together: $mg\sin\alpha - \frac{1}{2}mR\ddot\theta - m\ddot z = 0$

$z - R\theta = 0 \Rightarrow \ddot z = R\ddot\theta \Rightarrow mg\sin\alpha = \frac{3}{2}mR\ddot\theta$

$$\Rightarrow \ddot\theta = \frac{2g}{3R}\sin\alpha, \quad \ddot z = \frac{2g}{3}\sin\alpha, \quad \lambda = \frac{-gm}{3}\sin\alpha$$

Conservation Laws

Having worked some examples to demonstrate the utility of Lagrangian mechanics, now let's explore some of its deeper implications.

First, consider the E-L equation

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0.$$

If the Lagrangian is independent of a particular q_i, then that q_i is called a <u>cyclic</u> or <u>ignorable</u> coordinate. In this case,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i = \text{constant}.$$

p_i is called the <u>generalized momentum</u> conjugate to q_i.

Ex) One particle in potential that is independent of x

$$\mathcal{L} = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(y,z)$$

$$\Rightarrow p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \text{is conserved}$$

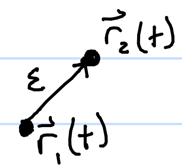
Noether's theorem

A fundamental fact in modern physics is that <u>symmetries</u> of a system imply <u>conservation laws</u>. We can verify this fact w/ Lagrangian mechanics.

Ex) Invariance under translation

If we take a system of N particles described by $\vec{r}_i(t)$ and shift every particle by a small amount $\vec{\varepsilon}$, certainly the kinetic energy will be the same.

Equivalently, $\delta T = 0$.



In particular, assume $U(\vec{r}_1, \ldots, \vec{r}_N) = U(\vec{r}_1 + \vec{\varepsilon}, \ldots, \vec{r}_N + \vec{\varepsilon})$

Equivalently, this says $\delta U = 0$. Then,

$\delta L = \delta T - \delta U = 0.$

Imagine that we have chosen $\vec{\varepsilon} = \varepsilon\hat{x}$. Then

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_1}\delta x_1 + \cdots + \frac{\partial \mathcal{L}}{\partial x_N}\delta x_N$$
$$= \frac{\partial \mathcal{L}}{\partial x_1}\varepsilon + \cdots + \frac{\partial \mathcal{L}}{\partial x_N}\varepsilon$$
$$= 0$$

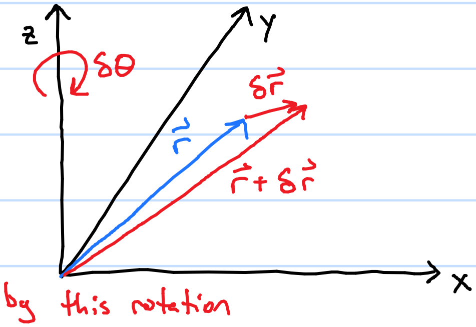
Thus, $\sum_i \frac{\partial \mathcal{L}}{\partial x_i} = 0$. From E-L, this implies

$$\sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = 0 = \sum_i \frac{d}{dt}(p_{ix}) = 0$$

$$\Rightarrow \sum_i p_{ix} = P_x = 0 \Leftrightarrow \text{total linear momentum is conserved}$$

Ex) Invariance under rotation

Imagine rotating a system about the e.g. z-axis by an amount $\delta\theta$.



Then: $\vec{r} \to \vec{r} + \delta\vec{\theta}\times\vec{r}$

Note: T will not be effected by this rotation

This rotation $\delta\vec{\theta}$ is "fixed", so $\dot{\vec{r}} \to \dot{\vec{r}} + \delta\vec{\theta}\times\dot{\vec{r}}$.

In our variational notation, and for $i=1,\dots,N$ particles,

$$\delta\vec{r}_i = \delta\vec{\theta}\times\vec{r}_i \quad \text{and} \quad \delta\dot{\vec{r}}_i = \delta\vec{\theta}\times\dot{\vec{r}}_i.$$

As before, if the system is invariant to rotations, then equivalently $\delta L = 0$:

$$\delta L = \sum_{\underset{1,\dots,N}{i},\ \underset{x,y,z}{a}} \frac{\partial L}{\partial x_{ai}}\delta x_{ai} + \underbrace{\frac{\partial L}{\partial \dot{x}_{ai}}}_{\text{(gen'l) momentum}}\delta\dot{x}_{ai} = 0.$$

But E-L $\Rightarrow \dfrac{\partial L}{\partial x_{ai}} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}_{ai}}\right) = \dfrac{d}{dt}\left(p_{ai}\right)$

$$\Rightarrow \delta L = 0 = \sum_{i,\alpha}\left(\frac{d}{dt}p_{ai}\right)\delta x_{ai} + p_{ai}\,\delta\dot{x}_{ai}$$

$$= \sum_i \dot{\vec{p}}_i\cdot\delta\vec{r}_i + \vec{p}_i\cdot\delta\dot{\vec{r}}_i$$

"triple products"

$$= \sum_i \dot{\vec{p}}_i\cdot\left(\delta\vec{\theta}\times\vec{r}_i\right) + \vec{p}_i\cdot\left(\delta\vec{\theta}\times\dot{\vec{r}}_i\right)$$

$$= \sum_i \delta\vec{\theta}\cdot\left[\vec{r}_i\times\dot{\vec{p}}_i + \dot{\vec{r}}_i\times\vec{p}_i\right]$$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

Thus,

$$\delta L = 0 = \delta\vec{\theta}\cdot\sum_i \frac{d}{dt}\left(\vec{r}_i\times\vec{p}_i\right)$$

$$= \delta\vec{\theta}\cdot\frac{d}{dt}\left(\sum_i \vec{r}_i\times\vec{p}_i\right)$$

for arbitrary small rotation $\delta\vec{\theta}$. Hence,

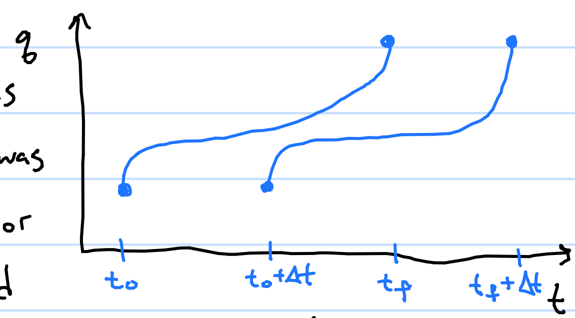
$$\boxed{\sum_i \vec{r}_i\times\vec{p}_i = \text{constant} = \sum_i \vec{L}_i = \vec{L}}$$

ang. mom.

Thus, rotational invariance $\Rightarrow$ consv. angular momentum

Ex) Invariance under time translation

A system that behaves identically whether it was started at time t_0 or time $t_0+\Delta t$ is called "time invariant," and will satisfy $\frac{\partial L}{\partial t} = 0$, e.g. there is no explicit time dependence. However, even when $\frac{\partial L}{\partial t} = 0$, in general $\frac{d}{dt}L \neq 0$:



$$\frac{d}{dt}L(q_1,\dots,q_s,\dot{q}_1,\dots,\dot{q}_N,t)$$

$$= \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}$$

by E-L $= \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = \dfrac{d}{dt}(p)$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\dot{q}_i\right) + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial L}{\partial t} = \sum_i \frac{d}{dt}\left(p_i\dot{q}_i\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\mathcal{L} = \frac{d}{dt}\left(\sum_i p_i \dot{q}_i\right) + \frac{\partial \mathcal{L}}{\partial t}.$$

Then, if $\frac{\partial \mathcal{L}}{\partial t} = 0$, $\sum_i p_i \dot{q}_i - \mathcal{L} = \text{const}$.

This conserved quantity is so important, we give it a special name: the Hamiltonian, $\mathcal{H}$

we will eventually learn Hamiltonian mechanics

$$\mathcal{H} := \sum_i p_i \dot{q}_i - \mathcal{L}$$

Whereas the Lagranian was given by $\mathcal{L} = T - U$, we will show that the Hamiltonian $\mathcal{H} = T + U$.

The kinetic energy T takes the form $T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2}$. We are going to rewrite this in gen'l coords.

$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_s, t)$. We are going to assume time-invariance $\Rightarrow$ $\vec{r}_i = \vec{r}_i(q_1, \ldots, q_n)$. Then

$$\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j \Rightarrow \dot{\vec{r}}_i^{\,2} = \sum_j \sum_k \left(\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right)$$

Thus,
$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i \sum_{j,k}\left(\frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right)$$
$$\equiv \frac{1}{2}\sum_{j,k} A_{jk}\dot{q}_j\dot{q}_k, \quad \text{with} \quad A_{jk} \equiv \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j}\cdot\frac{\partial \vec{r}_i}{\partial q_k}$$

Then
$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j A_{ij}\dot{q}_j$$

it is obvious if $S = 1$ $\Rightarrow T = \frac{1}{2}\dot{q}^2 \Rightarrow \frac{\partial T}{\partial \dot{q}} = \dot{q}$

this is not obvious but you will prove it on your HW

Then, $\sum_i p_i \dot{q}_i = \sum_i \sum_j A_{ij}\dot{q}_i\dot{q}_j = 2T$

Thus,

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = 2T - (T - U)$$
$$\Rightarrow \boxed{\mathcal{H} = T + U}$$

More broadly, this says that if the Lagrangian is time invariant $\left(\frac{\partial \mathcal{L}}{\partial t} = 0\right)$, then the Hamiltonian — the total energy of the system — is conserved!

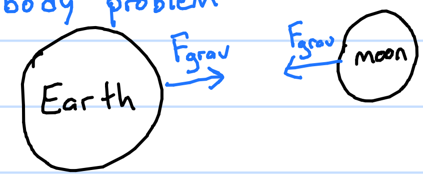
"homogeneity of space"
Translation invariance $\Rightarrow$ linear momentum conserved
"isotropy of space"
Rotation invariance $\Rightarrow$ angular momentum conserved
Time translation invariance $\Rightarrow$ energy conserved

These statements are realizations of Noether's theorem, which says that in general, symmetries in the Lagrangian imply conserved quantities. This theorem has ramifications throughout all of physics!!

Chapter 8: Two-body Central-force Problems

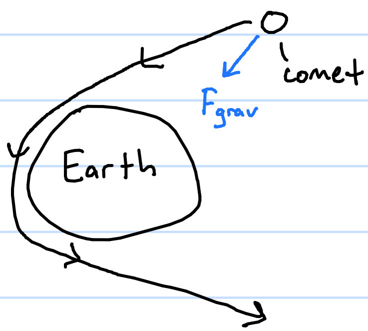
Next we will apply Lagrangian mechanics to some of the most fundamental problems in physics: central-force problems.

"two-body problem"

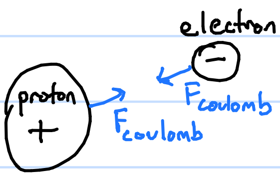


$$U_{grav} = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

Kepler problem

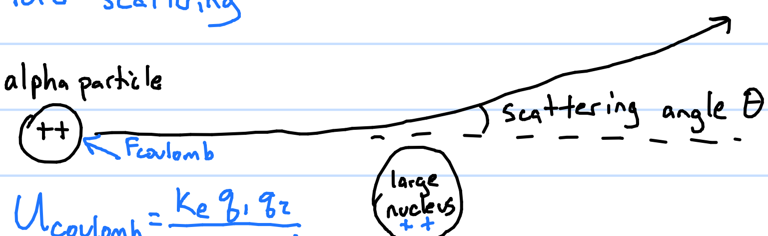


Hydrogen atom



$$U_{coulomb} = \frac{k_e q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

Rutherford scattering



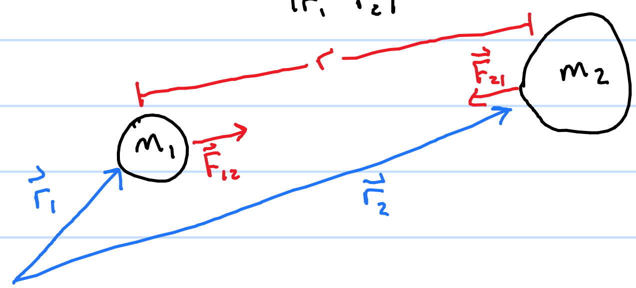
$$U_{coulomb} = \frac{k_e q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

While you will likely study these problems in detail in future classes, we will cover some of the important basics today.

Consider two objects w/ masses m_1 and m_2, and assume the forces of their interaction are conservative and central, i.e. they have a potential energy function $U(\vec{r}_1, \vec{r}_2) = \underbrace{U(|\vec{r}_1 - \vec{r}_2|)}_{\text{this is a "central force"}}$,

e.g. $U = U_{grav} = \frac{-G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$ or

$U = U_{coulomb} = \frac{k_e q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$.

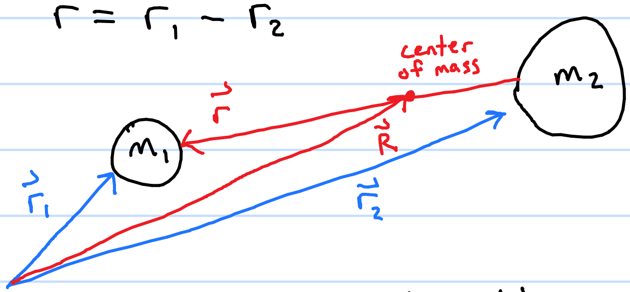


Once we introduce $r \equiv |\vec{r}_1 - \vec{r}_2|$, we may write $\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(r)$, and analyze this Lagrangian to understand the system.

The variables r_1 and r_2 will prove to be somewhat unwieldy, so we will introduce new gen'l variables: the center of mass $\vec{R}$, and the relative coordinate $\vec{r}$:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{\underbrace{M}_{\text{total mass } m_1+m_2}}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$



We can now find $\mathcal{L} = \mathcal{L}(\vec{r}, \vec{R})$:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \tfrac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2 - U(|\vec{r}|)$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \tfrac{1}{2}\left(\frac{m_1 m_2^2}{M^2} + \frac{m_1^2 m_2}{M^2}\right)\dot{\vec{r}}^2 - U(|\vec{r}|)$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}\frac{m_1 m_2}{M}\dot{\vec{r}}^2 - U(|\vec{r}|)$$

It is common to introduce the reduced mass

$$\mu = \frac{m_1 m_2}{M} = \frac{m_1 m_2}{m_1+m_2} \quad \text{so that}$$

$$\mathcal{L} = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\mathcal{L}_{COM}} + \underbrace{\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(|\vec{r}|)}_{\mathcal{L}_{relative}}$$

Now we may immediately put our conservation results from the previous section to work.

Notice $\vec{R}$ is an "ignorable coordinate":

$$\frac{\partial\mathcal{L}}{\partial\vec{R}} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\vec{R}}}\right) = 0$$

$$\Rightarrow M\dot{\vec{R}} = \text{constant}$$

$\Rightarrow$ center of mass of system moves w/ constant velocity. We choose a center-of-mass reference frame, so that in this frame $\dot{\vec{R}} = 0$.

In this frame $\mathcal{L}_{COM} = 0$, and

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - U(|\vec{r}|) \Rightarrow \mu\ddot{\vec{r}} = -\nabla U(|\vec{r}|)$$

Thus, with ignorable coordinates we have converted a two-body problem into a one-body problem! But we can simplify it even further.

Consider the angular momentum, which is conserved:

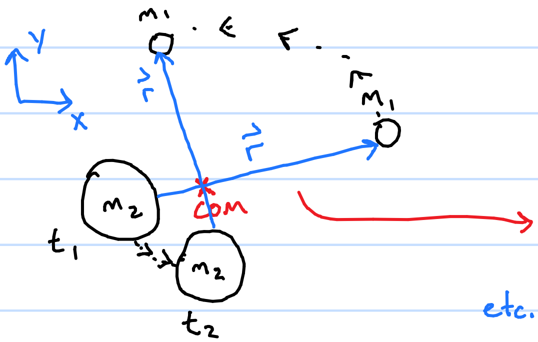
$$\vec{L} = \vec{r}_1\times\vec{p}_1 + \vec{r}_2\times\vec{p}_2 = m_1\vec{r}_1\times\dot{\vec{r}}_1 + m_2\vec{r}_2\times\dot{\vec{r}}_2. \quad \text{In COM } \vec{r}_1 = \frac{m_2}{M}\vec{r}, \ \vec{r}_2 = \frac{-m_1}{M}\vec{r}$$

$$\Rightarrow \vec{L} = \left[\frac{m_1 m_2^2}{M^2} + \frac{m_1^2 m_2}{M^2}\right](\vec{r}\times\dot{\vec{r}})$$

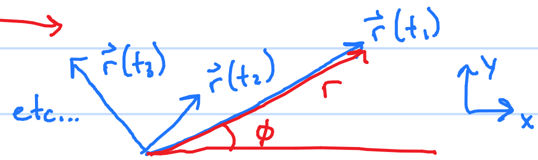
$$= \frac{m_1 m_2}{M}(\vec{r}\times\dot{\vec{r}}) \Rightarrow \vec{L} = \mu\vec{r}\times\dot{\vec{r}} = \text{const.}$$

Thus $\vec{r}$ and $\dot{\vec{r}}$ must always be parallel to a constant vector. This requires the entire motion of $\vec{r}$ to remain in a plane. Thus, we only have 2 DOFs, not 3.

Thus we may represent two-body motion as a one-body system in a plane:



If we only are tracking $\vec{r}$ (i.e. fixing the base of each vector):



Convenient generalized coordinates to parameterize $\vec{r}$ are (r, ϕ). Then:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Notice that ϕ is ignorable, i.e. $\frac{\partial \mathcal{L}}{\partial \phi} = 0$. Thus

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \underbrace{\mu r^2 \dot{\phi}}_{\equiv \ell_z} = \text{constant}$$

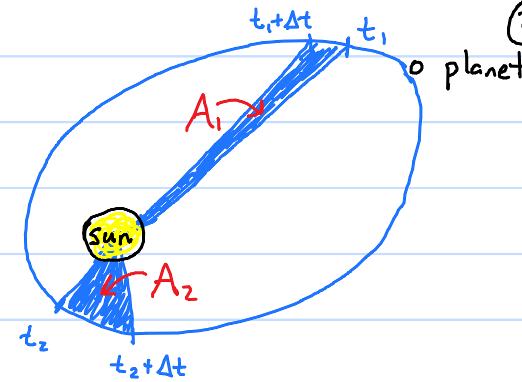
$\equiv \ell_z$, angular momentum in $\hat{z}$ direction (since we assume motion occurs in xy-plane)

Thus, the angular momentum of the relative coordinate $\vec{r}$ is conserved as well.

We can use this knowledge to prove Kepler's second law, from way back in 1609.

Kepler's second law:

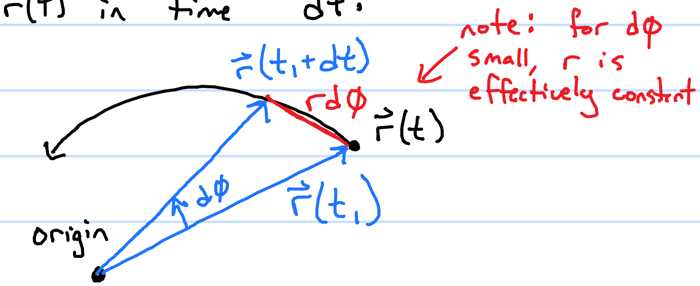
"A line joining a planet and the sun sweeps out equal areas during equal intervals of time."



To prove this, consider the differential area swept out by $\vec{r}(t)$ in time dt:

$$dA = \frac{1}{2} r^2 d\phi$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi}$$



But we just proved $\ell_z = \mu r^2 \dot{\phi}$ = constant.

$$\Rightarrow \frac{dA}{dt} = \frac{\ell_z}{2\mu} = \text{constant}$$

The other E-L equation for r, called the <u>radial equation</u>, is:

centrifugal force

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \Rightarrow \mu\ddot{r} = \mu r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

But we know $\dot{\phi} = \frac{\ell}{\mu r^2}$ = const, so substitute it in:

$$\mu\ddot{r} = \frac{\mu r \ell^2}{\mu^2 r^4} - \frac{\partial U}{\partial r} \Rightarrow \mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$$

Finally, we can express the centrifugal force $F_{cf} = \frac{\ell^2}{\mu r^3}$ in terms of a potential U_{cf}:

$$F_{cf} = -\frac{\partial}{\partial r} U_{cf} \Rightarrow U_{cf} = \frac{\ell^2}{2\mu r^2}$$

Now we can write our radial equation as

$$\mu\ddot{r} = -\frac{dU_{eff}}{dr} = -\frac{d}{dr}\left[U_{cf} + U(r)\right]$$

"effective" potential

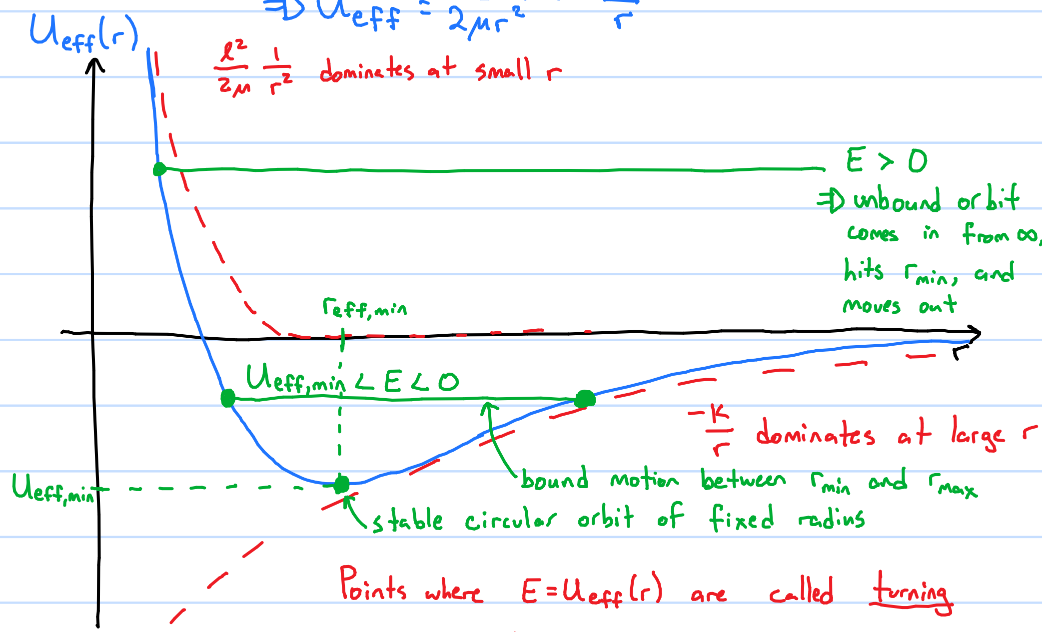
Thus, we have transformed the two-body problem into the problem of a 1-dimensional particle in a potential U_{eff}.

We can use another conservation result from the previous section: there is no explicit time dependence $\left(\frac{\partial \mathcal{L}}{\partial t} = 0\right)$. Thus the energy (aka Hamiltonian) is conserved:

$$E = T + U = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r)$$
$$= \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$
$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + U_{cf} + U(r)$$
$$\Rightarrow E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

We can use this potential energy U_{eff} to conceptually understand the different behaviors the system may exhibit. In particular focus on gravitational attraction $\Rightarrow U = \frac{-k}{r}$ w/ $k = Gm_1m_2$.

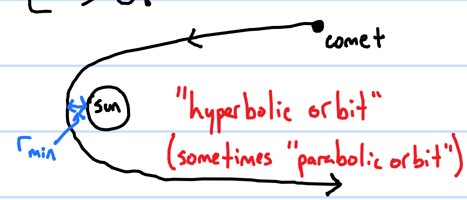
$$\Rightarrow U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r}$$



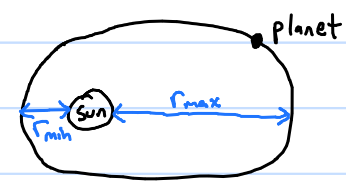
Points where $E = U_{eff}(r)$ are called turning points, labeled r_{min} and r_{max}. They satisfy $\dot{r} = 0$.

What do these orbits look like?

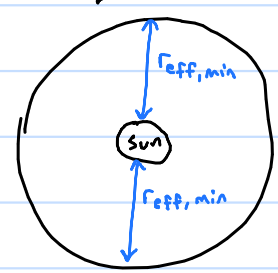
E > 0:



$U_{eff,min} < E < 0$:



$E = U_{eff,min}$:



"circular orbit"

Now that we have an intuition for what these orbits look like, let's compute them explicitly.

From before, we have $\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{\partial U}{\partial r}$, with $U(r) = -\frac{k}{r}$ and $k = G m_1 m_2$. Thus, $\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{k}{r^2}$.

This problem requires two tricks:

1) Use $u = \frac{1}{r}$. This kind of makes sense, since $r_{min} < r < \infty$ becomes $0 < u < \frac{1}{r_{min}}$, so now our variable is bounded.

2) Notice that for $r = r(\phi)$, $\frac{dr}{dt} = \frac{dr}{d\phi}\dot{\phi}$.

Now, consider $\dot{r} = \frac{dr}{dt} = \dot{\phi}\frac{dr}{d\phi}$. But we already found $\dot{\phi} = \frac{\ell}{\mu r^2} = \text{const}$. Then

$$\dot{r} = \frac{\ell}{\mu r^2}\frac{dr}{d\phi}.$$ Now use $r = \frac{1}{u}$ to see

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = \frac{\ell u^2}{\mu}\left(\frac{-1}{u^2}\frac{du}{d\phi}\right) = \frac{-\ell}{\mu}\frac{du}{d\phi}$$

Plug this into $\ddot{r}$ to find

$$\ddot{r} = \frac{d}{dt}\dot{r} = \dot{\phi}\frac{d}{d\phi}(\dot{r}) = \frac{\ell}{\mu r^2}\left(\frac{-\ell}{\mu}\frac{d^2u}{d\phi^2}\right) = \frac{-\ell^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2}$$

And finally we can rewrite the E-L eqns as

$$\mu\ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{k}{r^2} \Rightarrow \frac{-\ell^2 u^2}{\mu}\frac{d^2u}{d\phi^2} = u^2\left(\frac{\ell^2}{\mu}u - k\right)$$

$$\Rightarrow \boxed{\frac{d^2u}{d\phi^2} = -u + \frac{\mu k}{\ell^2}}$$ radial equation in $u = \frac{1}{r}$ variable

How to solve $u'' = -u + k\mu/\ell^2$?

Choose $w = u - k\mu/\ell^2$. Then

$$w'' = -w \Rightarrow w = A\cos(\phi - \phi_0)$$

$\phi_0 \to 0$: we can set ϕ_0 to zero by changing where $\phi = 0$ is

A can be forced to be positive by choosing $\phi_0 = \phi_0 + \pi$

$$\Rightarrow u = A\cos(\phi) + \frac{k\mu}{\ell^2}$$

$$:= \frac{k\mu}{\ell^2}(1 + \epsilon\cos\phi), \text{ where } \underbrace{\epsilon}_{\text{positive constant}} = \frac{A\ell^2}{k\mu}$$

Thus, $\frac{1}{r} = \frac{k\mu}{\ell^2}(1 + \epsilon\cos\phi)$

$$\Rightarrow r(\phi) = \frac{\ell^2/(k\mu)}{1 + \epsilon\cos\phi}$$

A few notes immediately from this form:

→ if $\epsilon < 1$ the denominator is never zero
$\Rightarrow r(\phi)$ is <u>bounded</u> (elliptical orbit)

→ if $\epsilon \geq 1$ the denominator becomes zero for some ϕ
$\Rightarrow r(\phi)$ is <u>unbounded</u> (hyperbolic orbit)

For an elliptical orbit, $r_{min} = r(0)$ and $r_{max} = r(\pi)$

$$\Rightarrow r_{min} = \frac{\ell^2/k\mu}{1+\epsilon}, \quad r_{max} = \frac{\ell^2/k\mu}{1-\epsilon}$$

For an unbounded orbit, $r_{min} = \frac{\ell^2/k\mu}{1+\epsilon}$, too.

With a Lagrangian approach, we used gen'l coords and symmetries to reduce the two-body problem into an analytically tractable 1-dimensional problem.

Phys 104 Midterm Review

The midterm will cover Taylor chapters 6, 7, 8, and 9.1 - 9.2. I recommend reading my lecture notes, reviewing discussion section WUS, reading the posted homework solutions, and reading the textbook to prepare for the test.

Concepts you should be familiar with:

- Calculus of variations (aka how to extremize functionals). Euler-Lagrange equations and their derivation.

  functional $I[y] = \int_{x_1}^{x_2} f(y, y', x)\, dx$ extremized if

  $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$ ... Euler-Lagrange equation

- Lagrangian mechanics. Hamilton's principle. Action $S = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)\, dt$. Lagrangian $\mathcal{L} = T - U$. Principle of stationary action. Principle of virtual work at static equilibrium $\delta W = \sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_i = 0$. D'Alembert's principle $\sum_{i=1}^{N} (\vec{F}_i^{(a)} - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$. Virtual displacements. Variational notation e.g. $\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)\right] \delta q$. Constraints and constraint equations $\phi(q_1, \ldots, q_s) = 0$. E-L eqns w/ constraints $\frac{\partial \mathcal{L}}{\partial q_i} + \lambda(t) \frac{\partial \phi}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0$. Constraint force $\lambda(t) \frac{\partial \phi}{\partial q_i}$. Conservation laws (ignorable coordinates, time $\rightarrow$ energy, translation $\rightarrow$ lin. momentum, rotation $\rightarrow$ ang. mom). Hamiltonian $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L} = E$.

- Two-body central-force problems. Central force $U(|\vec{r}_1 - \vec{r}_2|)$. Kepler problem. Hydrogen atom. Center of mass coordinate $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$. Relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$. Reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$. Conservation of angular momentum + energy. Effective potential energy and associated hyperbolic and ellipsoidal orbits.

- Noninertial reference frames. Observers in inertial frames S_0 versus noninertial frames S. "Fictitious" inertial forces to "fix" Newton's 2nd law in noninertial frames $m\ddot{\vec{r}} = \vec{F} + \vec{F}_{inertial}$. Inertial force $\vec{F}_{inertial} = -m\vec{A}$.

Flow chart (approximate)

Problem statement $\longrightarrow$ Draw a picture

Problem statement $\xrightarrow{\text{care about constraint forces}}$ Write down constraint eqn $\longrightarrow$ Choose generalized coords

Problem statement $\xrightarrow{\text{don't care about constraint forces}}$ Choose generalized coords

Choose generalized coords $\xrightarrow{\text{static equilibrium}}$ Principle of virtual work

Choose generalized coords $\xrightarrow{\text{care about dynamics}}$ Write down Lagrangian (or modified Lagrangian w/ constraint)

$\downarrow$

Evaluate symmetries: ignorable coords, explicit time dependance, invariance under translations / rotations

$\downarrow$ simplify problem w/ symmetries, if possible

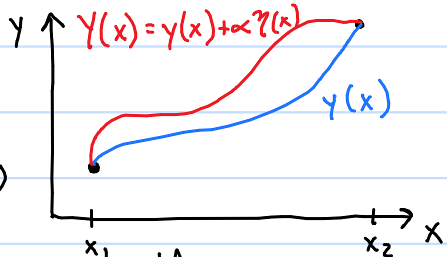
Write down Lagrange's eqns (aka E-L eqns)

↳ Solve for eqns of motion

↳ Interpret eqns, take limits, hone intuition

ex) Derivation of E-L eqns

Goal: minimize functional $A[y] = \int_{x_1}^{x_2} f(y, y', x)\,dx$.



Consider a perturbed trajectory $Y(x) = y(x) + \alpha\eta(x)$, requiring $\eta(x_1) = \eta(x_2) = 0$. Then

$A[y] = A[\alpha; y]$ is extremized if $\frac{dA}{d\alpha} = 0$ at $\alpha = 0$. So,

$$\frac{dA}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f(Y,Y',x)}{\partial Y}\underbrace{\frac{\partial Y}{\partial \alpha}}_{\eta(x)} + \frac{\partial f(Y,Y',x)}{\partial Y'}\underbrace{\frac{\partial Y'}{\partial x}}_{\eta'(x)}\,dx$$

Use integration by parts to see

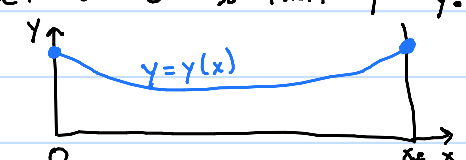
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)\,dx = \underbrace{\left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_1}^{x_2}}_{\text{0 at boundaries}} - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta(x)\,dx.$$ Thus

$$\frac{dA}{d\alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx = 0.$$

But $\eta(x)$ was arbitrary, so we require

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

for $y(x)$ to extremize A, where we set $\alpha = 0$ so that $Y \to y$.



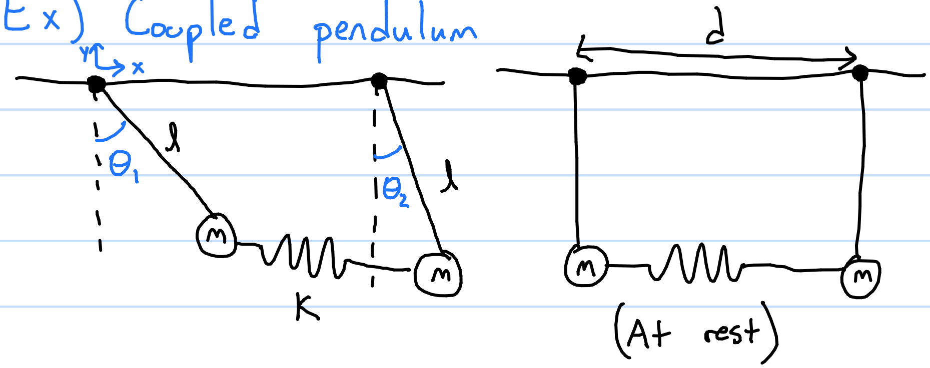
Ex) Hanging rope

Goal: minimize potential energy of rope $U = mgh$.

Locally $dm = \rho\,ds = \rho\sqrt{dx^2 + dy^2} = \rho\sqrt{1 + y'^2}\,dx$ (ρ: density of rope)

$$\Rightarrow U = \rho g \int_0^{x_f} y\sqrt{1 + y'^2}\,dx$$ $\Rightarrow$ Use E-L w/ $f(y, y', x) = y\sqrt{1 + y'^2}$

Ex) Coupled pendulum



Two pendulums (length ℓ, mass m) are coupled by a spring (spring constant k, rest length d).

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m\dot{\vec{r}}_2^{\,2} - mgy_1 - mgy_2 - \frac{1}{2}k\left[|\vec{r}_2 - \vec{r}_1| - d\right]^2$$

$\vec{r}_1 = (\ell\sin\theta_1, -\ell\cos\theta_1)$, $\quad \vec{r}_2 = (d + \ell\sin\theta_2, -\ell\cos\theta_2)$

$\Rightarrow \dot{\vec{r}}_1^{\,2} = \ell^2\dot{\theta}_1^2$, $\quad \dot{\vec{r}}_2^{\,2} = \ell^2\dot{\theta}_2^2$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\ell^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + mg\ell\left(\cos\theta_1 + \cos\theta_2\right) - \frac{1}{2}k\left(\sqrt{(d + \ell\sin\theta_2 - \ell\sin\theta_1)^2 + (\ell\cos\theta_2 - \ell\cos\theta_1)^2} - d\right)^2$$

In the small θ_1, θ_2 limit $\ell\cos\theta_2 - \ell\cos\theta_1 \approx 0$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m\ell^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + mg\ell\left(\cos\theta_1 + \cos\theta_2\right) - \frac{k}{2}\left(\ell\sin\theta_2 - \ell\sin\theta_1\right)^2$$

$$\Rightarrow m\ell^2\ddot{\theta}_1 = -mg\ell\sin\theta_1 + k\left(\ell\sin\theta_2 - \ell\sin\theta_1\right)\ell\cos\theta_1$$

$$m\ell^2\ddot{\theta}_2 = -mg\ell\sin\theta_2 - k\left(\ell\sin\theta_2 - \ell\sin\theta_1\right)\ell\cos\theta_2$$

Also for small θ_1, θ_2, $\sin\theta \approx \theta$, $\cos\theta = 1 + O(\theta^2)$

$$\Rightarrow m\ell^2\ddot{\theta}_1 = -mg\ell\,\theta_1 + k\ell^2(\theta_2 - \theta_1)$$

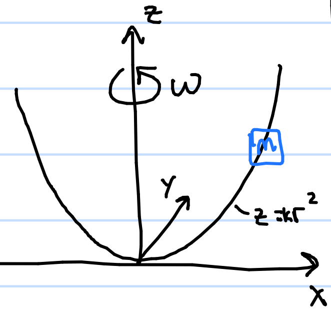
$$m\ell^2\ddot{\theta}_2 = -mg\ell\,\theta_2 - k\ell^2(\theta_2 - \theta_1).$$ Linear!

Rewrite $\ddot{\vec{\theta}} = A\vec{\theta}$

$\Rightarrow$ "normal modes", which we will learn later.

Ex) Bead on spinning parabola

A bead of mass m is constrained to be on a wire parabola spinning at angular velocity ω, with gravity.



Symmetry suggests cylindrical coords (r, ϕ, z).

Gen'l coord? $q = r$; ϕ and z are constrained:

$$\phi(t) = \phi_0 + \omega t \ ; \quad z = kr^2$$

3 DOF − 2 constraints = 1 gen'l coord, r

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2). \quad \dot{z} = 2kr\dot{r}, \quad \dot{\phi} = \omega$$

$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\omega^2 + 4k^2r^2\dot{r}^2)$$

$$U = mgz = mgkr^2$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\omega^2 + 4k^2r^2\dot{r}^2) - mgkr^2$$

$$\Rightarrow \frac{d}{dt}(m\dot{r} + 4mk^2r^2\dot{r}) = mr\omega^2 + 4mk^2r\dot{r}^2 - 2mgkr$$

$$\Rightarrow \cancel{m}\ddot{r} + 8\cancel{m}k^2r\dot{r}^2 + 4\cancel{m}k^2r^2\ddot{r} = \cancel{m}r\omega^2 + 4\cancel{m}k^2r\dot{r}^2 - 2\cancel{m}gkr$$

$$\Rightarrow \ddot{r}(1 + 4k^2r^2) + 4k^2r\dot{r}^2 = r(\omega^2 - 2gk)$$

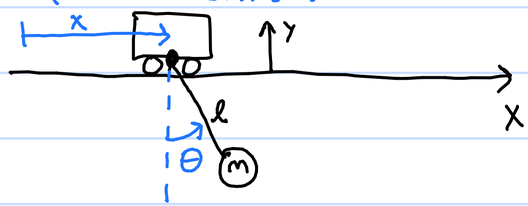
Note $r = 0 \Rightarrow \ddot{r} = 0$, so fixed point.

If $r = \varepsilon \ll 1$, $\dot{r} = 0$, what happens?

$\ddot{r} = r(\omega^2 - 2gk) \Rightarrow$ if $\omega > \sqrt{2gk}$, $\ddot{r} > 0 \Rightarrow$ unstable

if $\omega < \sqrt{2gk}$, $\ddot{r} < 0 \Rightarrow$ stable

Ex) Pendulum on a cart (from discussion)



The cart undergoes driven oscillations so that

$x_{origin} = A\cos\omega t \quad \Rightarrow \quad x = A\cos\omega t + l\sin\theta \Rightarrow \dot{x} = -A\omega\sin\omega t + l\cos\theta\,\dot{\theta}$

$y = -l\cos\theta \Rightarrow \dot{y} = l\sin\theta\,\dot{\theta}$

$$\Rightarrow T = \tfrac{1}{2}m\left[A^2\omega^2\sin^2\omega t - 2A\omega l\sin\omega t\cos\theta\,\dot{\theta} + \underline{l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\theta}^2}\right]$$

$$\Rightarrow T = \tfrac{1}{2}m\left[A^2\omega^2\sin^2\omega t - 2A\omega l\sin\omega t\cos\theta\,\dot{\theta} + l^2\dot{\theta}^2\right]$$

$$U = mgh = -mgl\cos\theta$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m\left[A^2\omega^2\sin^2\omega t - 2A\omega l\sin\omega t\cos\theta\,\dot{\theta} + l^2\dot{\theta}^2\right] + mgl\cos\theta$$

$$\frac{d}{dt}\left[-mA\omega l\sin\omega t\cos\theta + ml^2\dot{\theta}\right] = ml^2\ddot{\theta} - mA\omega^2 l\cos\omega t\cos\theta \underline{+ mA\omega l\sin\omega t\sin\theta\,\dot{\theta}}$$

$$\frac{\partial}{\partial\theta}\mathcal{L} = \underline{mA\omega l\sin\omega t\sin\theta\,\dot{\theta}} - mgl\sin\theta$$

$$\Rightarrow ml^2\ddot{\theta} - mA\omega^2 l\cos\omega t\cos\theta + mgl\sin\theta = 0$$

$$\Rightarrow \ddot{\theta} = \frac{A\omega^2}{l}\cos\omega t\cos\theta - \frac{g}{l}\sin\theta \quad \rightarrow \text{reduces to } \ddot{\theta} = -\frac{g}{l}\sin\theta \text{ if } A = 0$$

What if we are in noninertial frame moving w/ cart?

$$\vec{F}_{inertial} = -m\vec{A}. \quad \vec{A} = \frac{d^2}{dt^2}(A\cos\omega t\,\hat{i}) = -A\omega^2\cos\omega t\,\hat{i}$$

$$\Rightarrow \vec{F}_{inertial} = mA\omega^2\cos\omega t\,\hat{i} = \frac{-\partial U_{inertial}}{\partial x} \Rightarrow U_{inertial} = -mA\omega^2\cos\omega t\; x$$

$$\Rightarrow U_{inertial} = -mA\omega^2\cos\omega t\; l\sin\theta$$

$$\Rightarrow \mathcal{L} = T - (U + U_{inertial})$$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2}m(l^2\dot{\theta}^2) + mgl\cos\theta + mA\omega^2 l\cos\omega t\sin\theta$$

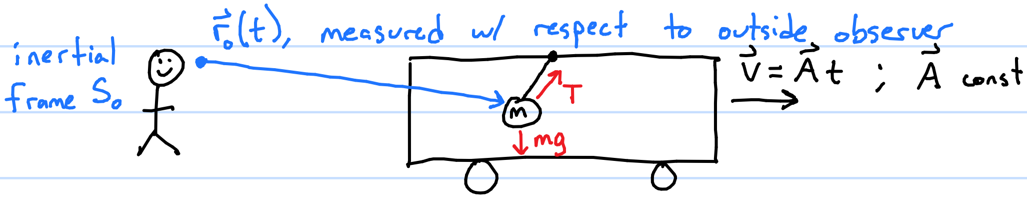
$$\Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta + mA\omega^2 l\cos\omega t\cos\theta$$

SAME AS BEFORE!

Chapter 9: Mechanics in noninertial frames

Often we are interested in understanding and viewing physics in a noninertial reference frame — for example, what systems look like while riding in an accelerating train, or when seated on a rotating turn-table, or when standing on the rotating Earth.

Consider a pendulum on an accelerating train, as viewed by an inertial observer:

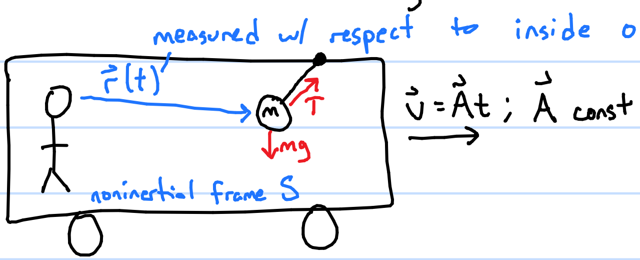


Newton's second law: $m\ddot{\vec{r}}_0 = \vec{F} = \vec{T} + m\vec{g}$

acceleration to right means $\vec{F}$ must point to right $\Rightarrow$ crooked pendulum

$\Rightarrow$ As an inertial observer, everything works!

What if we are riding in the train?



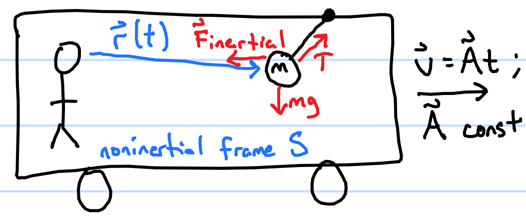
In the accelerating frame S, Newton's 2^{nd} law says:

$m\ddot{\vec{r}} = \vec{F}$

at equilibrium, to inside observer $\ddot{\vec{r}} = 0$. But $\vec{F}$ has horizontal component!

$\Rightarrow$ As a noninertial observer, something is broken.

As a noninertial observer, we can fix this by adding a fictitious force $\vec{F}_{inertial}$, that is added solely to make N2L work in a noninertial reference frame:



in S:

$m\ddot{\vec{r}} = \vec{F} = \vec{T} + \vec{F}_{inertial} + m\vec{g}$

What is $\vec{F}_{inertial}$? We can relate N2L in S and S_0 with $\dot{\vec{r}}_0 = \dot{\vec{r}} + \vec{V}$ ← velocity of train

($\dot{\vec{r}}_0$: inertial observer; $\dot{\vec{r}}$: noninertial observer)

Then $\ddot{\vec{r}}_0 = \ddot{\vec{r}} + \vec{A} \Rightarrow m\ddot{\vec{r}}_0 = m\ddot{\vec{r}} + m\vec{A}$.

Use the two forms of N2L:

$\Rightarrow \vec{T} + m\vec{g} = \vec{T} + m\vec{g} + \vec{F}_{inertial} + m\vec{A}$

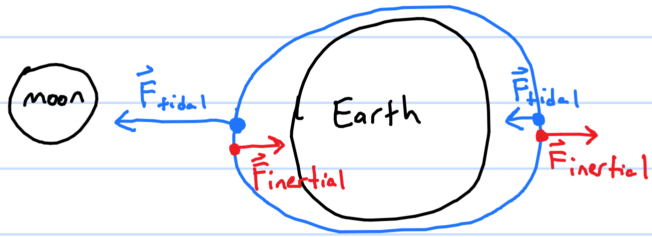
$\Rightarrow \vec{F}_{inertial} = -m\vec{A}$

N2L in noninertial frame: $\boxed{m\ddot{\vec{r}} = \vec{F} + \vec{F}_{inertial}}$

You feel $\vec{F}_{inertial}$ when slamming the brakes in your car, or turning around a bend, or accelerating from rest.

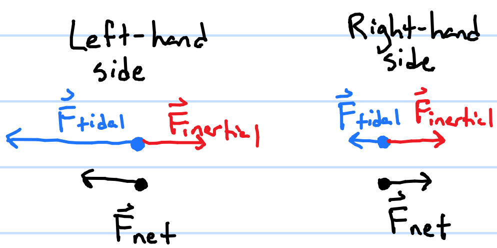
Thus, we can "fix" N2L if we are in a linearly accelerating reference frame. Later we will do the same thing to "fix" N2L in a rotating frame.

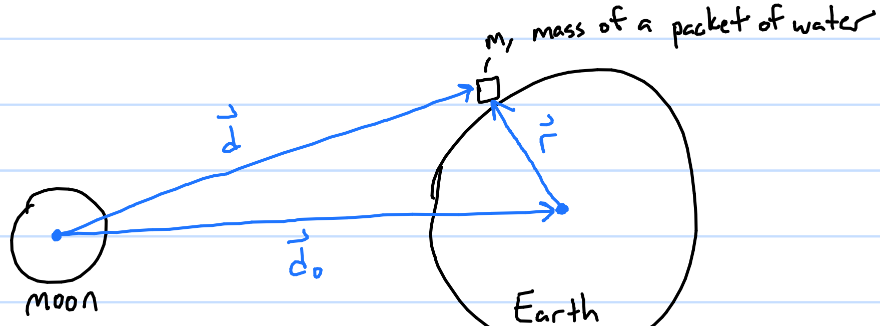
Ex) Why are there two tides per day?



* for this problem we assume the Earth is entirely water *

Conceptual understanding: $\vec{F}_{inertial}$ due to centripetal acceleration of Earth about moon is constant (approximately); $\vec{F}_{tidal}$ due to gravitational attraction is larger when closer to the moon.





The acceleration the Earth feels towards the moon is due to gravitation:

$$F_{grav} = \frac{-G m_{moon} M_{Earth}}{|\vec{d}_0|^2}\hat{d}_0 = M_{Earth}\vec{A}_{Earth}$$

$$\Rightarrow \vec{A}_{Earth} = \frac{-G m_{moon}}{|\vec{d}_0|^2}\hat{d}_0$$

If we consider how our packet of water will accelerate, $\vec{A} \approx \frac{-G m_{moon}}{|\vec{d}_0 + \vec{r}|^2}\hat{d}_0$

$$\sim \frac{-G m_{moon}}{|\vec{d}_0|^2}\frac{1}{1+\frac{r^2}{d_0^2}} \sim \frac{-G m_{moon}}{|\vec{d}_0|^2}\Big(1 - \underbrace{2\frac{r^2}{d_0^2}}_{\text{very small}}\Big)$$

Thus we may approximate $\vec{A}_{water} \approx \vec{A}_{Earth}$.

Then, we can write N2L for a noninertial frame:

$$m\ddot{\vec{r}} = \vec{F} - m\vec{A} \qquad d \equiv |\vec{d}|,\ d_0 \equiv |\vec{d}_0|$$

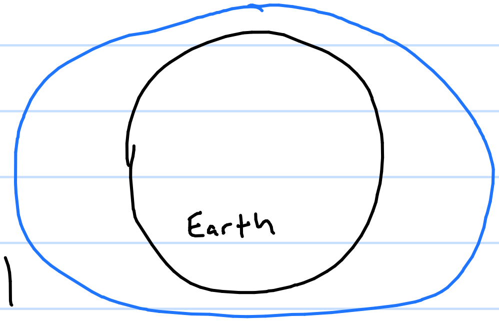
$$= \underbrace{m\vec{g}}_{\text{Earth's gravity}} \underbrace{- \frac{G m_m m}{d^2}\hat{d}}_{\text{moon's gravity}} + \underbrace{\vec{F}_{ng}}_{\text{other forces (e.g. buoyancy)}} + \underbrace{\frac{G m_m m}{d_0^2}\hat{d}_0}_{\text{inertial force}}$$

$$\Rightarrow m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{tidal} + \vec{F}_{ng}, \quad w/$$

$$\vec{F}_{tidal} = -G m_m m\left(\frac{\hat{d}}{d^2} - \frac{\hat{d}_0}{d_0^2}\right)$$

Thus, if $d < d_0$, $\vec{F}_{tidal}$ points towards the moon $(-\hat{d})$;
if $d > d_0$, $\vec{F}_{tidal}$ points away from the moon $(+\hat{d})$

Additionally, at "north pole" x-components cancel almost exactly, so $\vec{F}_{tidal}$ points south. At south pole, $\vec{F}_{tidal}$ points north.

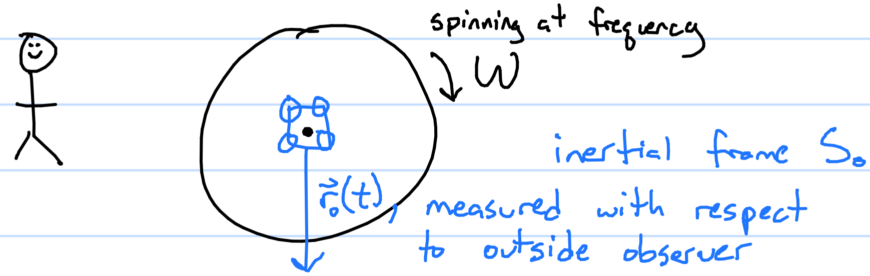


To find the actual heights of these tides, the argument is that the difference in potential energies at low and high tides due to Earth's gravity ($= mg\Delta h$) must balance the difference in potential energies due to the moon's gravity (roughly, $= U_{tidal}(\text{high point}) - U_{tidal}(\text{low point})$, where $U_{tidal} = -GM_m m\left(\frac{1}{d} + \frac{x}{d_0^2}\right)$). This calculation is performed in full in the book, Eqs. (9.13) – (9.18). The upshot is that the moon induces a tide of ≈ 54cm, and the sun induces one of 25cm. These tides may interfere constructively or destructively.

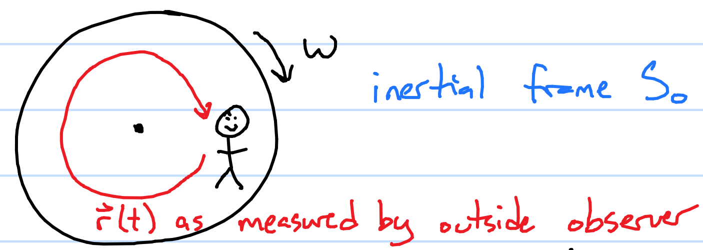
Next we will examine rotating noninertial reference frames. Once again, we will seek to "fix" N2L so that it works in this frame.

Consider a giant rotating turn-table that you can stand on, if you wish. Also consider a flying drone that starts at the center and flies in a straight line to the edge.

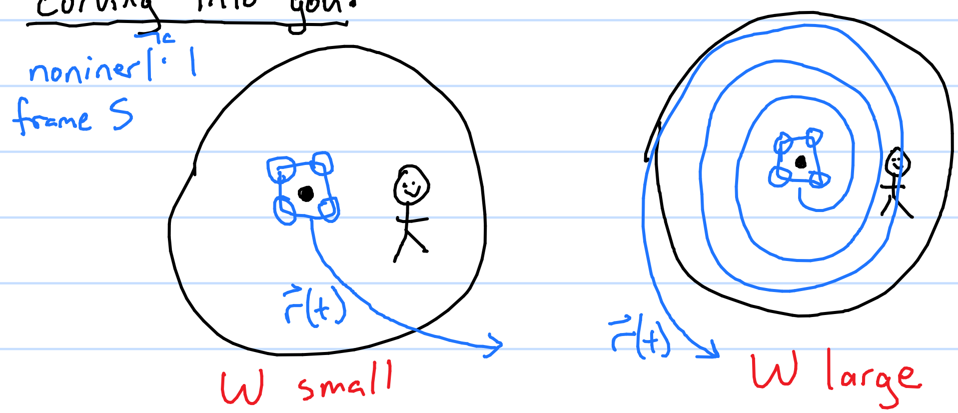
If you are not on the turn table, you see a drone fly in a straight line:



If you are on the turn table, to an inertial observer you look like:

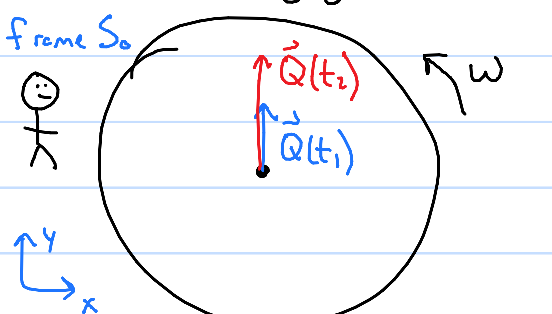


But now if the drone flies straight down again, to you it will seem as if it is curving into you:



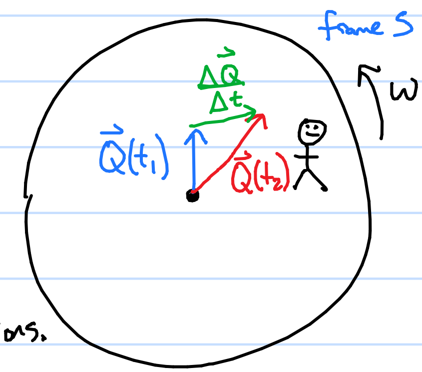
Thus, in the rotating frame it is as if a force is acting on the drone to move it. In rotating reference frames, these inertial forces are the centrifugal and Coriolis.

In this turn table example, we have already seen that straight lines in S_0 are not straight lines in S_0. Consider again a turn table, but this time imagine a vector $\vec{Q}(t)$ changing on it:



notation for derivative in frame S_0

That is, $\left(\frac{d\vec{Q}}{dt}\right)_{S_0}$ is in the $\hat{y}$ direction.



If we are on the turn table:

That is, $\left(\frac{d\vec{Q}}{dt}\right)_S$ has components in both the x and y directions.

Thus, we need to relate

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \text{some inertial term}$$

How do we find what this inertial term is?

Having questioned our intuition to determine conceptually what should happen, we now turn to mathematics in order to be precise.

Consider three orthonormal vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ that are fixed in the rotating frame S. To observers in the inertial frame S_0, these three vectors are rotating. In S then,

$$\vec{Q} = Q_1\hat{e}_1 + Q_2\hat{e}_2 + Q_3\hat{e}_3 = \sum_{i=1}^{3} Q_i\hat{e}_i.$$

In S, Q is changing while the $\hat{e}_i$ stay fixed:

$$\left(\frac{d\vec{Q}}{dt}\right)_S = \sum_{i=1}^{3} \frac{dQ_i}{dt}\hat{e}_i$$

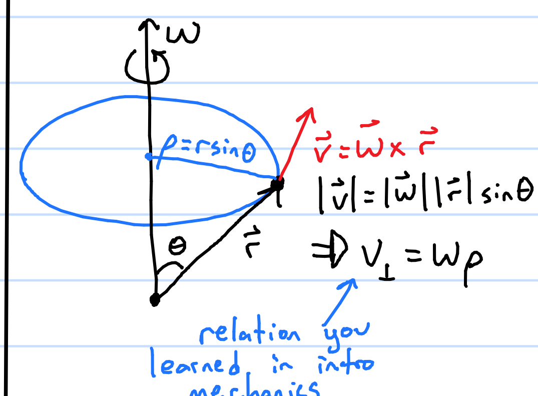
In S_0, Q is changing and the $\hat{e}_i$ are changing:

$$\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_{i=1}^{3} \frac{dQ_i}{dt}\hat{e}_i + \sum_{i=1}^{3} Q_i\frac{d\hat{e}_i}{dt}$$

What does $\frac{d\hat{e}_i}{dt}$ mean? More generally consider how any vector $\vec{r}$ changes due to rotation:

remember the right-hand rule!

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$



In particular, applying this relation to $\hat{e}_i$ yields

$$\frac{d\hat{e}_i}{dt} = \vec{\omega} \times \hat{e}_i$$

Thus, $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \sum_{i=1}^{3}\left[\frac{dQ_i}{dt}\hat{e}_i + \vec{\omega} \times (Q_i\hat{e}_i)\right]$, and so

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\omega} \times \vec{Q}}$$

Relation for time derivatives in inertial vs. noninertial frames.

In particular, we may notice

$$\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \left(\frac{d\hat{e}_i}{dt}\right)_S + \vec{\omega} \times \hat{e}_i,$$

but the $\hat{e}_i$ were chosen to be fixed in S, so

$$\left(\frac{d\hat{e}_i}{dt}\right)_{S_0} = \vec{\omega} \times \hat{e}_i.$$

Thus the velocity of the $\hat{e}_i$ rotates and points in the plane of rotation.

Do we need to worry about $\vec{\omega}$ on Earth?

Kind of. Earth rotates 2π every 24 hours:

$$\omega = \frac{2\pi \text{ radians}}{24 \times 3600\,s} = 7.3e^{-5} \frac{rad}{s}.$$

Thus, ω plays a role when $\vec{Q}$ is very big (i.e. when it represents the velocity of a rocket), or when measurements of $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} - \left(\frac{d\vec{Q}}{dt}\right)_S$ become very precise.

Let's use our newfound rotational machinery to "fix" N2L in a rotating noninertial frame.

Inertially, $m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$.

Thus, we must find $\left(\frac{d^2\vec{r}}{dt}\right)_{S_0}$ in terms of the rotating frame.

First, $\left(\frac{d\vec{r}}{dt}\right)_{S_0} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\omega} \times \vec{r}.$

To take a second derivative, let

$\vec{A} = \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\omega} \times \vec{r}$ and consider:

$$\left(\frac{d\vec{A}}{dt}\right)_{S_0} = \left(\frac{d\vec{A}}{dt}\right)_S + \vec{\omega} \times \vec{A}$$

$$\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d^2\vec{r}}{dt^2}\right)_S + \underbrace{\left(\frac{d}{dt}\right)_S \vec{\omega} \times \vec{r}}_{\text{use product rule w/ } \frac{d\vec{\omega}}{dt} = 0} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\Rightarrow \left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \left(\frac{d^2\vec{r}}{dt^2}\right)_S + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_S + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

note $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Use N2L to replace $m\left(\frac{d^2\vec{r}}{dt^2}\right)_{S_0} = \vec{F}$ and rearrange:

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_S = \vec{F} + 2m\left(\frac{d\vec{r}}{dt}\right)_S \times \vec{\omega} + m(\vec{\omega} \times \vec{r}) \times \vec{\omega},$$

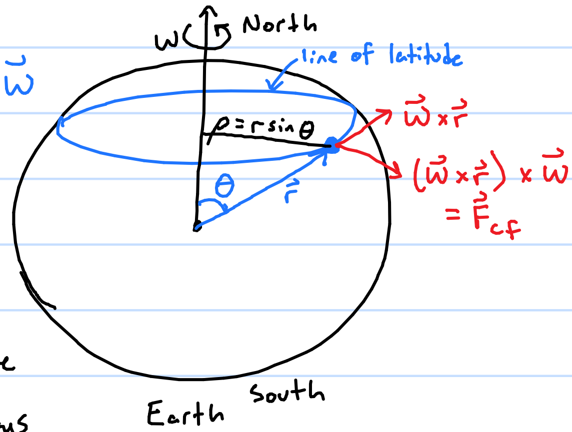
or using the notation that all derivatives are in S, we arrive at N2L in a rotating noninertial reference frame:

$$\boxed{m\ddot{\vec{r}} = \vec{F} + \underbrace{2m\,\dot{\vec{r}} \times \vec{\omega}}_{\substack{\text{Coriolis force} \\ \vec{F}_{cor}}} + \underbrace{m(\vec{\omega} \times \vec{r}) \times \vec{\omega}}_{\substack{\text{Centrifugal force} \\ \vec{F}_{cf}}}}$$

Notice in terms of magnitudes, $\vec{F}_{cor} \sim m\dot{v}\omega$ (important for very fast things) and $\vec{F}_{cp} \sim m\omega^2 r$.

Let's explore the centrifugal force first, therefore assuming $v \ll \omega r$ to neglect the Coriolis force.

$$\vec{F}_{cf} = m(\vec{\omega} \times \vec{r}) \times \vec{\omega}$$



Note both $(\vec{\omega}\times\vec{r})$ and $(\vec{\omega}\times\vec{r})\times\vec{\omega}$ are in the plane of rotation, but $(\vec{\omega}\times\vec{r})$ is tangent to the path and $(\vec{\omega}\times\vec{r})\times\vec{\omega}$ always points radially outwards from the axis of rotation.

Thus, $\vec{F}_{cf} = m\omega^2\rho\,\hat{\rho}$, if we define $\hat{\rho}$ to be the same $\hat{\rho}$ as in cylindrical coordinates. (Note: take $v = \omega\rho$ to see $\vec{F}_{cf} = mv^2/\rho$, as we have seen in introductory mechanics.

Even freefall, as observed by an observer on Earth, is a little complicated:

Earth mass   Earth radius

$$m\ddot{\vec{r}} = \vec{F}_{grav} + \vec{F}_{cf} = \underbrace{-\frac{GMm}{R^2}\hat{r}}_{\equiv m\vec{g}_0} + \vec{F}_{cf}$$

Plugging in $\vec{F}_{cf}$ from before,

$$m\ddot{\vec{r}} = m\vec{g}_0 + m\omega^2 R\sin\theta\,\hat{\rho},$$

where we use $r \approx R$ since the Earth's radius is huge compared to any height we would drop from.

To compare these two terms we need to take the $-\hat{r}$ component of the $\hat{\rho}$ term.

Use $\rho\hat{\rho}\cdot(-\hat{r}) = -\rho\cos(\frac{\pi}{2}-\theta) = -\rho\sin\theta$

$$F_{rad} = (\vec{F}_{grav} + \vec{F}_{cf})\cdot(-\hat{r}) = \left(\frac{GMm}{R^2} - m\omega^2 R\sin^2\theta\right)$$

$$\Rightarrow g_{rad} = \underbrace{\frac{GM}{R^2}}_{9.8\,m/s^2} - \underline{\omega^2 R\sin^2\theta}$$

at equator $\theta = \pi$

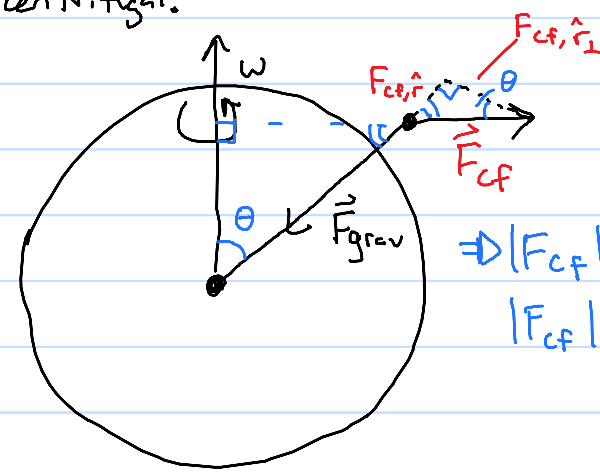
at poles $\theta = 0$

Thus, at the equator gravity differs by

$$\omega^2 R = (7.3\cdot10^{-5}\,s^{-1})^2(6.4\times10^6\,m) \approx 0.034\,m/s^2$$

This difference can be measured!

There is an additional tangential acceleration due to the centrifugal.



in $\hat{r}$ dir

$$\Rightarrow |F_{cf}|\sin\theta = F_{cf,\hat{r}}$$

$$|F_{cf}|\cos\theta = F_{cf,\hat{r}_\perp}$$

in direction orthogonal to $\hat{r}$

Tangential gravitational acceleration: $g_{tang} = \frac{|F_{cf}|\cos\theta}{m}$

$\Rightarrow g_{tang} = \omega^2 R\sin\theta\cos\theta$. At max $(\theta = \frac{\pi}{4})$,

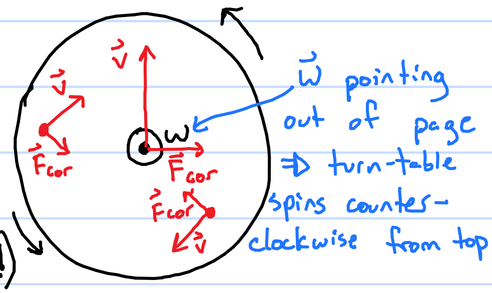
$g_{tang} = \frac{1}{2}\omega^2 R \approx .017\,m/s^2$. But this is very hard to measure! (Since hard to know where $\vec{g}$ points exactly.)

Let us now turn to the Coriolis force,

$$\vec{F}_{cor} = 2m\dot{\vec{r}}\times\vec{\omega}$$

← compare to $\vec{F} = q\vec{v}\times\vec{B}$ in EM

To think about this force, we return to the turn-table:



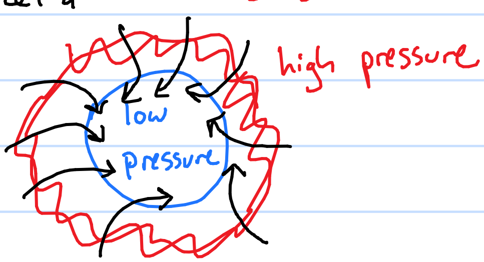
As seen in this figure, $\vec{F}_{cor}$ always works to deflect the velocity <u>to the right</u>.

(If $\vec{\omega}\to-\vec{\omega}$, this would be flipped)

Now imagine this turn table is the <u>north pole</u>!

Suppose you have a low pressure region:

Initially, all the air from the high pressure region will rush into the low pressure region. But it will all feel a <u>rightward deflection</u>:



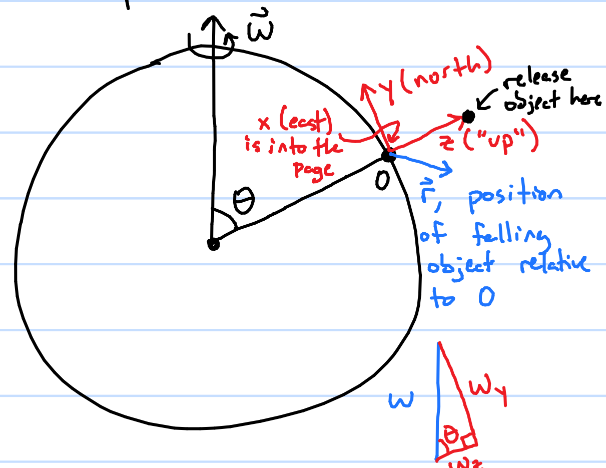
This generates a counterclockwise circulation that, if sufficiently violent, is a hurricane/cyclone/typhoon.

Now we'll return to free fall on the Earth's surface, but this time we won't neglect Coriolis forces:

$$m\ddot{\vec{r}} = \underbrace{m\vec{g}_0}_{\text{due to } U_{grav}} + \overbrace{\vec{F}_{cf}}^{\text{from before}} + \vec{F}_{cor} = m\vec{g} + \vec{F}_{cor},$$

where we have used $\vec{g}$ to represent the net gravitational and centrifugal forces. Then $m\ddot{\vec{r}} = m\vec{g} + 2m\dot{\vec{r}}\times\vec{\omega}$.

It is important to choose intuitive coordinates:



z… up (in the direction of $-\vec{g}$)

y… north (along line of longitude)

x… east (along line of latitude)

Note in these coordinates $\vec{\omega} = \omega\sin\theta\,\hat{y} + \omega\cos\theta\,\hat{z}$.

Also, $\dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$. Thus,

$$\dot{\vec{r}}\times\vec{\omega} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \dot{x} & \dot{y} & \dot{z}\\ 0 & \omega\sin\theta & \omega\cos\theta\end{vmatrix} = \hat{x}\left(\dot{y}\omega\cos\theta - \dot{z}\omega\sin\theta\right) - \hat{y}\left(\dot{x}\omega\cos\theta\right) + \hat{z}\left(\dot{x}\omega\sin\theta\right)$$

All together, $m\ddot{\vec{r}} = m\vec{g} + 2m\dot{\vec{r}}\times\vec{\omega}$ implies

$$\ddot{x} = 2\dot{y}\omega\cos\theta - 2\dot{z}\omega\sin\theta$$
$$\ddot{y} = -2\dot{x}\omega\cos\theta$$
$$\ddot{z} = 2\dot{x}\omega\sin\theta - g$$

These are the <u>true</u> equations of motion for an object in free-fall.

In the limit $\omega \to 0$, these reduce to $\ddot{x} = \ddot{y} = 0$, $\ddot{z} = -g$ as usual, leading to

$$\ddot{x} = 2\dot{y}\omega\cos\theta - 2\dot{z}\omega\sin\theta$$
$$\ddot{y} = -2\dot{x}\omega\cos\theta \qquad (*)$$
$$\ddot{z} = 2\dot{x}\omega\sin\theta - g$$

$\underbrace{x = y = 0, \quad z = h - \tfrac{1}{2}gt^2}_{\text{"zeroth order approximation"}}$ (assuming $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = 0$, and $z(0) = h$)

$(= \omega^0 = 1$

Now we will use a neat mathematical trick: substitute our zeroth order approximation into our exact equation $(*)$ to improve our approximation: notice $\ddot{x}$ is a fn of $\dot{z}$:

$$\ddot{x} = -2(-gt)\omega\sin\theta \qquad x(t) = \tfrac{1}{3}gt^3\omega\sin\theta$$
$$\ddot{y} = 0 \qquad \Rightarrow \qquad y(t) = 0$$
$$\ddot{z} = -g \qquad z(t) = h - \tfrac{1}{2}gt^2$$

This is called a "first order" $\underbrace{(\omega^1 = \omega)}_{\text{first order in } \omega}$ approximation.

In your HW you extend this to a second order approximation. The upshot is that now the object moves in the x direction (east)!

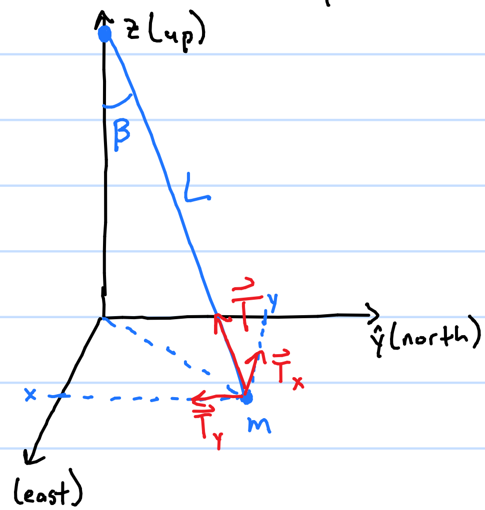
By how much? $z = 0$ at $t = \sqrt{2h/g}$. Imagine a 100m mine shaft at the equator:

$$\Rightarrow x\left(\sqrt{2h/g}\right) = \tfrac{1}{3}g\left(\sqrt{2h/g}\right)^3 \omega \sin\left(\tfrac{\pi}{2}\right) = \tfrac{1}{3}\left(10\tfrac{m}{s}\right)\left(20\ s^2\right)^{3/2}\left(7.3e\text{-}5\ s^{-1}\right)$$
$$\approx 2.2\ \text{cm}$$

This is measurable! (At north pole $\overset{\theta = 0}{\text{there}}$ would be no such deflection)

ex) Foucalt pendulum

We may measure the influence of this Coriolis force explicitly via the Foucalt pendulum:



This is just a pendulum free to swing in 3D with a heavy mass and a light + long wire connected to the ceiling.

As before,

$$m\ddot{\vec{r}} = \vec{T} + \underbrace{m\vec{g}_0 + \vec{F}_{cf}}_{\equiv m\vec{g}} + \vec{F}_{cor}$$

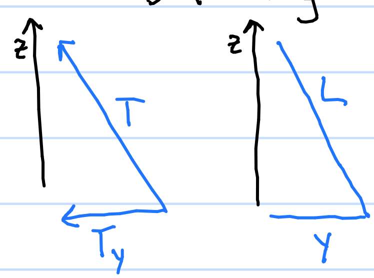
$$\Rightarrow m\ddot{\vec{r}} = \vec{T} + m\vec{g} + 2m\underline{\dot{\vec{r}} \times \vec{\omega}}$$

this term causes the pendulum to twist: constant clockwise force in northern hemisphere

We will assume the angle β is small, so that

$$T_z = mg = T\cos\beta \approx T(1 - \beta^2/2) \approx T \Rightarrow T \approx mg$$

To determine T_x and T_y, use similar triangles:



$$\left|\frac{T}{T_y}\right| = \left|\frac{L}{y}\right| \Rightarrow T_y = -T\frac{y}{L} = -\frac{mgy}{L}$$

$$\left|\frac{T}{T_x}\right| = \left|\frac{L}{x}\right| \Rightarrow T_x = -T\frac{x}{L} = -\frac{mgx}{L}$$

Then we can plug everything into N2L:

$$m\ddot{\vec{r}} = \vec{T} + m\vec{g} + \underbrace{2m\dot{\vec{r}}\times\vec{\omega}}$$

$= \hat{x}(\dot{y}\omega\cos\theta - \dot{z}\omega\sin\theta) - \hat{y}(\dot{x}\omega\cos\theta) + \hat{z}(\dot{x}\omega\sin\theta)$, from before

$$\Rightarrow \ddot{x} = \frac{-gx}{L} + 2\dot{y}\omega\cos\theta - 2\dot{z}\omega\sin\theta$$

$$\ddot{y} = \frac{-gy}{L} - 2\dot{x}\omega\cos\theta$$

$$\ddot{z} = mg - mg + \dot{x}\omega\sin\theta$$

We will focus on the pendulum's rotation, i.e. we don't care about $\ddot{z}$. Additionally, $\dot{z}$ is small relative to $\dot{x}$ and $\dot{y}$ for small oscillations, so

$$\ddot{x} = \frac{-g}{L}x + 2\dot{y}\omega\cos\theta$$

$$\ddot{y} = \frac{-g}{L}y - 2\dot{x}\omega\cos\theta.$$

Use $\frac{g}{L} = \omega_0^2$, the frequency of the pendulum's oscillation. Additionally, $\omega\cos\theta = \omega_z$. Hence

$$\ddot{x} - 2\omega_z\dot{y} + \omega_0^2 x = 0$$

$$(\ddot{y} + 2\omega_z\dot{x} + \omega_0^2 y = 0)\cdot i$$

To solve this, we use the trick $\eta = x + iy$, multiply the $\ddot{y}$ eqn by i, and add them together:

$$\ddot{x} + i\ddot{y} - 2\omega_z\underbrace{(\dot{y} - i\dot{x})}_{= -i(\dot{x}+i\dot{y})} + \omega_0^2(x+iy) = 0$$

$$\Rightarrow \ddot{\eta} + 2i\omega_z\dot{\eta} + \omega_0^2\eta = 0$$

We just need to solve this 2^{nd} order ODE. A clever guess ("ansatz") might be $\eta(t) = e^{-i\alpha t}$ for some α. Then,

$$\ddot{\eta} + 2i\omega_z\dot{\eta} + \omega_0^2\eta = (-i\alpha)^2\eta + 2i\omega_z(-i\alpha)\eta + \omega_0^2\eta = 0$$

$$= \eta\left[-\alpha^2 + 2\alpha\omega_z + \omega_0^2\right] = 0,$$

and so for our guess to work we require

$$\alpha = \frac{-2\omega_z \pm\sqrt{4\omega_z^2 + 4\omega_0^2}}{-2} \approx \omega_z \pm \omega_0$$

since $\omega_0 \gg \omega$
(pendulum) (earth)

Thus a solution to our ODE is

$$\eta(t) = C_1 e^{-i(\omega_z+\omega_0)t} + C_2 e^{-i(\omega_z-\omega_0)t}$$

$$= e^{-i\omega_z t}\left[C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t}\right]$$

Assume $\eta(0) = \underset{x_0}{A} + i\underset{y_0}{(0)}$, $\dot{\eta}(0) = 0$

0 since $\omega_z \ll \omega_0$

$$\Rightarrow C_1 + C_2 = A \;;\; \left[\cancel{-i\omega_z} e^{i\omega_z t}\left[C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t}\right] + e^{i\omega_z t}\left[-i\omega_0 C_1 e^{-i\omega_0 t} + i\omega_0 C_2 e^{i\omega_0 t}\right]\right]\Bigg|_{t=0} = 0$$

$$\Rightarrow -i\omega_0 C_1 + i\omega_0 C_2 = 0 \Rightarrow C_1 = C_2 \Rightarrow C_1 = C_2 = \frac{A}{2}$$

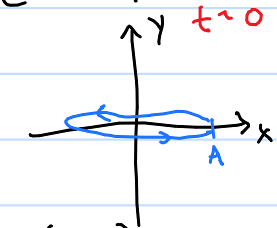
$$\Rightarrow \eta(t) = \frac{Ae^{-i\omega_z t}}{2}\left[e^{-i\omega_0 t} + e^{i\omega_0 t}\right] = Ae^{-i\omega_z t}\cos(\omega_0 t)$$

$$\Rightarrow \boxed{\eta(t) = Ae^{-i\omega_z t}\cos(\omega_0 t)}$$

$$\boxed{\eta(t) = A e^{-i\omega_z t} \cos(\omega_0 t)}$$

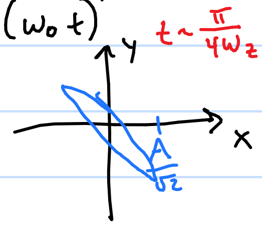
What does this mean, with $\eta = x + iy$, and $\omega_z \ll \omega_0$? At small times, $e^{-i\omega_z t} \approx 1$

$\Rightarrow x(t) = A\cos(\omega_0 t)$, $y(t) = 0$



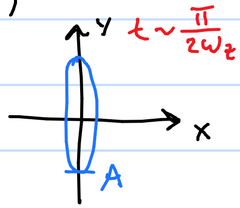
At $t = \frac{\pi}{4\omega_z}$, $e^{-i\omega_z t} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$\Rightarrow x(t) = \frac{A}{\sqrt{2}}\cos(\omega_0 t)$, $y(t) = -\frac{A}{\sqrt{2}}\cos(\omega_0 t)$



At $t = \frac{\pi}{2\omega_z}$, $e^{-i\omega_z t} = -i$

$\Rightarrow x(t) = 0$, $y(t) = -A\cos(\omega_0 t)$



How long is $t = \frac{\pi}{2\omega_z}$?

$\omega_z = \omega\cos\theta$, so in Santa Barbara ($\theta = 34°$)

$\Rightarrow \omega_z = .83\,\omega$

$= .83 \frac{2\pi \text{ radians}}{24 \times 3600 \text{ s}} = 6e-5 \frac{\text{rad}}{\text{s}} \Rightarrow \underline{t = 7 \text{ hours } 14 \text{ mins}}$

Thus, in ≈ 7 hours, a Foucault pendulum will rotate $90°$. Easily observed confirmation of Coriolis force!

Chapter 10: Rigid body rotation

In reality objects are not point masses, but exist in a continuum, with finite widths, and some geometry. We will consider one such object, the <u>rigid body</u>, which is a collection of N particles whose relative position to each other is fixed (aka rigid).

Akin to statistical mechanics, we will decompose this system of 6N DOF ($N \times$ (3 positions + 3 velocities)) into two subproblems: motion of the center of mass, and motion relative to this center.

So, consider N particles of position $\vec{r}_\alpha$, $\alpha = 1, \dots, N$. The center of mass position is:

$$\vec{R} = \sum_{i=1}^{N} \frac{m_i \vec{r}_i}{M},$$

or when the object is a continuous distribution of mass,

$$\vec{R} = \frac{1}{M} \int \vec{r}\, dm.$$

We know from our Lagrangian foray that $\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{\vec{R}}$, and therefore

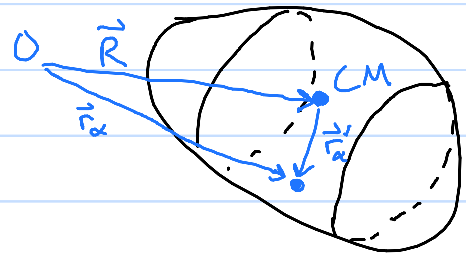
$\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{F}_{ext}$, i.e. that N2L works when we consider the collection of particles as a single lumped body. Thus, the center of mass of a rigid body moves like a point particle.

What about angular momentum? Consider a rigid body in the shape of an ellipsoid:

Define

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}'$$

some particle   relative position



Then the angular momentum of this particle about the origin is $\vec{\ell}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$

$$\Rightarrow \vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$$

$$= \sum_\alpha m_\alpha (\vec{R} + \vec{r}_\alpha{}') \times (\dot{\vec{R}} + \dot{\vec{r}}_\alpha{}')$$

$$= \sum_\alpha m_\alpha \vec{R} \times \dot{\vec{R}} + \sum m_\alpha \vec{R} \times \dot{\vec{r}}_\alpha{}' + \sum m_\alpha \vec{r}_\alpha{}' \times \dot{\vec{R}} + \sum m_\alpha \vec{r}_\alpha{}' \times \dot{\vec{r}}_\alpha{}'$$

$$= \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha{}' + \left(\sum_\alpha m_\alpha \vec{r}_\alpha{}'\right) \times \dot{\vec{R}} + \sum_\alpha \vec{r}_\alpha{}' \times m_\alpha \dot{\vec{r}}_\alpha{}'$$

$$\vec{L} = \vec{R} \times M\dot{\vec{R}} + \vec{R} \times \sum_\alpha m_\alpha \dot{\vec{r}}_{\alpha'} + \left(\sum_\alpha m_\alpha \vec{r}_{\alpha'}\right) \times \dot{\vec{R}}$$
$$+ \sum_\alpha \vec{r}_{\alpha'} \times m_\alpha \dot{\vec{r}}_{\alpha'}$$

$R = \frac{1}{M} \sum_\alpha m_\alpha r_\alpha$

$$= \sum_\alpha m_\alpha (\vec{r}_\alpha - \vec{R}) = \sum_\alpha m_\alpha r_\alpha - MR$$

But $\sum_\alpha m_\alpha \vec{r}_{\alpha'} = \vec{0}$, since the $\vec{r}_{\alpha'}$ are defined to be relative to the center of mass. Then,

$$\frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r}_{\alpha'}\right) = \sum_\alpha m_\alpha \dot{\vec{r}}_{\alpha'} = 0,$$ too. Hence

$$\vec{L} = \vec{R} \times M\dot{\vec{R}} + \sum_\alpha \vec{r}_{\alpha'} \times m_\alpha \dot{\vec{r}}_{\alpha'}$$
$$= \vec{L}_{com} + \vec{L}_{rel}$$

Being able to decompose physical quantities into CM and relative parts is a useful trick. For example, consider the kinetic energy,

$$T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}.$$

Break up $\vec{r}_\alpha$ into CM and relative terms so that

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha' \Rightarrow \dot{\vec{r}}_\alpha^{\,2} = \dot{\vec{R}}^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}_\alpha' + \dot{\vec{r}}_{\alpha'}^{\,2}.$$

Then,

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{R}}^2 + \dot{\vec{R}} \cdot \underbrace{\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'}_{=0,\text{ as above}} + \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'^{\,2}$$

$$\Rightarrow T = \underbrace{\frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{R}}^2}_{T_{cm}} + \underbrace{\frac{1}{2}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha'^{\,2}}_{T_{rel}}$$

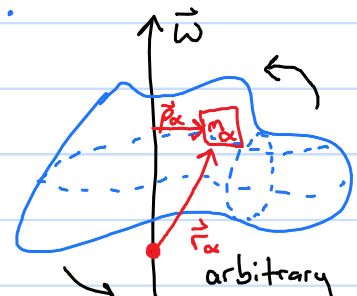
Further, for a rigid body, the only allowed relative motion is rotation. Then,

$$T = \underbrace{T_{\text{motion of CM}}}_{\text{looks like a point mass of mass } M} + T_{\text{rotation about CM}}$$

We have already extensively studied point masses ($T = \frac{1}{2} m \dot{\vec{r}}^2$), so understanding the CM dynamics is something we are familiar with. Thus, in the remainder of this chapter we will be studying rotations, and how to describe them.

Rotation about a fixed axis:

We will quantify the angular momentum $\vec{L}$ of a rotating rigid body, then use $\vec{L}$ to link rotations to the kinetic energy



arbitrary rigid body rotating about $\vec{\omega}$

(spoilers: $T_{rot} = \frac{1}{2} \vec{\omega} \cdot \vec{L}$) we can use this w/ our Lagrangian approach

Let $\vec{\omega}$ be in the $\hat{z}$-direction, i.e. $\vec{\omega} = \omega \hat{z}$.

The total angular momentum is $\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha$.

(Here we are <u>not</u> insisting these $\vec{r}_\alpha$ originate from the center of mass, but we <u>are</u> insisting they originate somewhere on the object's axis of rotation.)

Then, the velocity of each parcel is $\dot{\vec{r}}_\alpha = \vec{\omega} \times \vec{r}_\alpha$, with $\vec{r}_\alpha = x_\alpha \hat{x} + y_\alpha \hat{y} + z_\alpha \hat{z}$.

Then, $\vec{L} = \sum_\alpha \vec{r}_\alpha \times (m_\alpha \vec{\omega} \times \vec{r}_\alpha)$. Let's work component by component:

$$\vec{\omega} \times \vec{r}_\alpha = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = \underbrace{-\hat{x}\, y_\alpha \omega + \hat{y}\, x_\alpha \omega}$$

$$\vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -y_\alpha \omega & x_\alpha \omega & 0 \end{vmatrix} = -\hat{x}\, x_\alpha z_\alpha \omega - \hat{y}\, y_\alpha z_\alpha \omega + \hat{z}(x_\alpha^2 \omega + y_\alpha^2 \omega)$$

I_{zz} ... moment of inertia

$$\Rightarrow L_z = \sum_\alpha m_\alpha \omega (x_\alpha^2 + y_\alpha^2) \equiv I_{zz}\, \omega \qquad \underbrace{(x_\alpha^2 + y_\alpha^2)}_{\equiv \rho_\alpha^2}$$

$$I_{zz} \equiv \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

$$L_x = -\sum_\alpha m_\alpha \omega\, x_\alpha z_\alpha \equiv I_{xz}\, \omega, \quad I_{xz} \equiv -\sum_\alpha m_\alpha x_\alpha z_\alpha$$

$$L_y = -\sum_\alpha m_\alpha \omega\, y_\alpha z_\alpha \equiv I_{yz}\, \omega, \quad I_{yz} \equiv -\sum_\alpha m_\alpha y_\alpha z_\alpha$$

$$\Rightarrow \vec{L}_{(z)} = I_{xz}\omega\, \hat{x} + I_{yz}\omega\, \hat{y} + I_{zz}\omega\, \hat{z}$$

I_{xz}, I_{yz} ... products of inertia

component of $\vec{L}$ in $\hat{y}$-dir when rotating about $\hat{z}$

In introductory physics we learned $\vec{L} = I\vec{\omega}$, i.e. that $\vec{L}$ and $\vec{\omega}$ point in the same direction. Here we find that this is not necessarily the case, i.e. if $\vec{\omega}$ is in the $\hat{z}$-direction, there can exist components of $\vec{L}$ in the $\hat{x}$ and $\hat{y}$ directions.

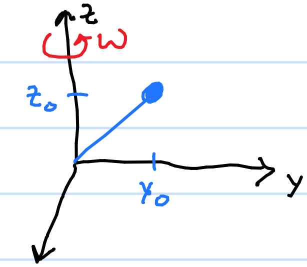
What determines this? Symmetry.

ex) Compute products and moment of inertia for point mass of mass m at $(0, y_0, z_0)$ rotating about $\hat{z}$.

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = 0$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = -m y_0 z_0$$

$$I_{zz} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2) = m(y_0^2 + z_0^2)$$

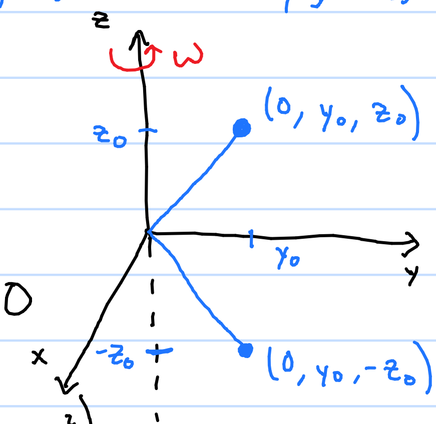


$$\Rightarrow \vec{L} = -m y_0 z_0 \omega\, \hat{y} + m(y_0^2 + z_0^2)\omega\, \hat{z}$$

$\Rightarrow$ component of angular momentum not in direction of axis of rotation

ex) Compute I_{xz}, I_{yz}, and I_{zz} for two point masses at $(0, y_0, z_0)$ and $(0, y_0, -z_0)$

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = -m(0 \cdot z_0 + 0 \cdot (-z_0)) = 0$$



$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = -m(y_0 z_0 + y_0(-z_0)) = 0$$

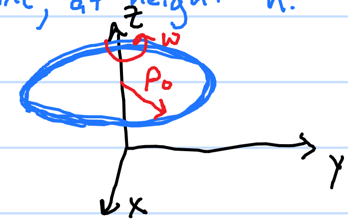
$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = 2m(y_0^2 + z_0^2)$$

$$\Rightarrow \vec{L} = 2m(y_0^2 + z_0^2)\omega\, \hat{z}$$

$\Rightarrow$ Symmetry $z \to -z$ implies $I_{xz} = I_{yz} = 0$

ex) Uniform ring of mass M and radius ρ_0, centered on z-axis and parallel to xy-plane, at height h.

No point masses $\Rightarrow$ need to integrate (but in this case we can use symmetry).



For each (x, z) there is $(-x, z)$ of equal mass $\Rightarrow I_{xz} = 0$

Likewise for $(y, z) \Rightarrow I_{yz} = 0$. $I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = M\rho_0^2$

It is straightforward to extend this machinery for arbitrary rotations $\vec{\omega}$, not just rotations about $\hat{z}$. Let $\vec{\omega} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$, and consider a body's angular momentum

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha),$$ and by distributivity

$$= \sum_\alpha m_\alpha \left[\vec{r}_\alpha \times (\vec{\omega}_x \times \vec{r}_\alpha) + \vec{r}_\alpha \times (\vec{\omega}_y \times \vec{r}_\alpha) + \vec{r}_\alpha \times (\vec{\omega}_z \times \vec{r}_\alpha) \right].$$

Each of these terms is similar to our previous case, and you can show

$$\vec{L} = \begin{bmatrix} I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{bmatrix},$$ where

$$I_{ab} = -\sum_\alpha m_\alpha a_\alpha b_\alpha,$$ with $a, b \in x, y, z$ and $a \neq b$ (e.g. I_{xy} or I_{zx})

$$I_{aa} = \sum_\alpha m_\alpha (b_\alpha^2 + c_\alpha^2),$$ with $a \in x, y, z$, and b and c being the other two directions

For example, $I_{zy} = -\sum_\alpha m_\alpha z_\alpha y_\alpha$, and

$I_{yy} = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2)$.

Then, we can write

$$\underset{\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix}}{\vec{L}} = \underset{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}{I} \overset{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}{\vec{\omega}}$$

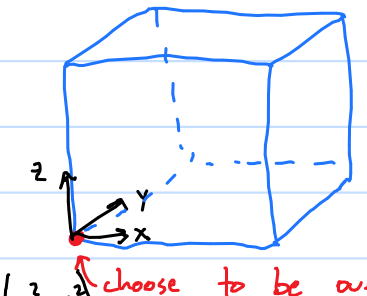
I ... inertia tensor

Note that $I_{ab} = I_{ba}$, that is, I is symmetric. Thus, to fully characterize the rotation of a system we only need to compute 6 ($I_{xy, xz, yz, xx, yy, zz}$) out of the 9 components of I.

Note that in $\vec{L} = I\vec{\omega}$, I depends on where the origin is. Thus, I will change depending on whether we rotate about e.g. the corner versus the edge of a body.

ex) Compute the inertia tensor for a cube of side a and mass M rotating about a corner.

Continuous object $\Rightarrow$ need to turn summations into integrals.



$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$
$$= \int dM (y^2 + z^2) = \int \rho \, dV \, (y^2 + z^2)$$

$M = \rho V \Rightarrow dM = \rho \, dV$. ρ = density, $\rho = \frac{M}{a^3}$

$$\Rightarrow I_{xx} = \int_0^a \int_0^a \int_0^a dx \, dy \, dz \; \rho (y^2 + z^2)$$

dx first; break up integral

$$= \rho a \int_0^a \int_0^a dy \, dz \, (y^2 + z^2) = \rho a \left[a \int_0^a dy \, y^2 + a \int_0^a dz \, z^2 \right]$$

$$= \rho a^2 \left[2 \frac{a^3}{3} \right] = \frac{2}{3} \rho a^5 = \frac{2}{3} \left[\frac{M}{a^3} \right] a^5$$

$$\Rightarrow I_{xx} = \frac{2}{3} M a^2$$

useful for simplifying calculations!

By symmetry, $I_{yy} = I_{zz} = I_{xx} = \frac{2}{3} M a^2$

$$I_{xy} = -\int_0^a \int_0^a \int_0^a dx \, dy \, dz \; \rho x y = -a\rho \int_0^a \int_0^a dx \, dy \; xy$$

$$= -a\rho \int_0^a dx \left[\frac{x y^2}{2} \right]_0^a = -a\rho \int_0^a dx \, \frac{a^2 x}{2} = \frac{-a^3 \rho}{2} \int_0^a dx \, x$$

$$= \frac{-a^5 \rho}{4} = \frac{-1}{4} a^5 \frac{M}{a^3} = \frac{-1}{4} M a^2.$$ By symmetry, all other off-diagonal terms of $I = \frac{-1}{4} M a^2$.

Then, in total, for a cube rotated about a corner,

$$I = \begin{bmatrix} q & r & r \\ r & q & r \\ r & r & q \end{bmatrix} \text{ with } q = \frac{2}{3}Ma^2 \text{ and } r = \frac{-1}{4}Ma^2$$

$$\Rightarrow I = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}.$$

If the cube is rotating about z, $\vec{\omega} = \omega\hat{z}$

$$\Rightarrow \vec{L} = I\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \Rightarrow \vec{L} = \frac{Ma^2\omega}{12}\left[-3\hat{x} - 3\hat{y} + 8\hat{z}\right].$$

Note $\vec{L}$ is not only in the $\hat{z}$-direction.

If $\vec{\omega}$ is along the main diagonal, $\vec{\omega} = \omega(\hat{x}+\hat{y}+\hat{z})$,

$$\vec{L} = I\begin{bmatrix} \omega \\ \omega \\ \omega \end{bmatrix} = \frac{Ma^2\omega}{12}\left[2\hat{x}+2\hat{y}+2\hat{z}\right] = \frac{Ma^2\omega}{6}(\hat{x}+\hat{y}+\hat{z}).$$

In this case, $\vec{L}$ is in the direction of $\vec{\omega}$. When this is the case, we say $\vec{\omega}$ is a principal axis, i.e. the angular momentum $\vec{L}$ due to a rotation about $\vec{\omega}$ is parallel to $\vec{\omega}$.

Mathematically, the condition that $\vec{\omega}$ is a principal axis is equivalent to $\vec{L} = \lambda\vec{\omega}$, where λ is a real number (not a matrix). Stated another way, this requires $\underset{\text{inertia tensor}}{I}\vec{\omega} = \underset{\text{real \#}}{\lambda}\vec{\omega}$ for a given rotation axis $\vec{\omega}$.

This statement $I\vec{\omega} = \lambda\vec{\omega}$ is an eigenvalue problem.

For the cube we found $\vec{\omega} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (it will be easier to use the normalized $\vec{\omega} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$) satisfied

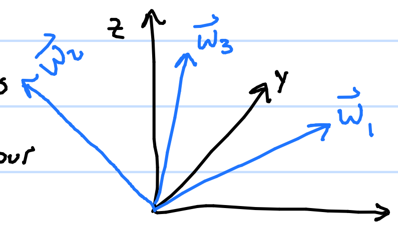
$$\vec{L} = I\vec{\omega} = \lambda\vec{\omega}, \text{ with } \lambda = Ma^2\omega/6.$$

$\vec{\omega} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ... eigenvector

$\lambda = Ma^2\omega/6$ ... eigenvalue

It is an established linear algebra fact that any eigenvalue equation $I\vec{\omega} = \lambda\vec{\omega}$, where I is a 3x3 symmetric matrix, will have 3 pairs $(\vec{\omega}_1,\lambda_1), (\vec{\omega}_2,\lambda_2), (\vec{\omega}_3,\lambda_3)$ of eigenvalues and eigenvectors. These eigenvectors are the axes of rotation that make $\vec{L}$ parallel to $\vec{\omega}$.

Thus, consider changing coordinates so that $\vec{\omega}_1, \vec{\omega}_2$, and $\vec{\omega}_3$ are our axes instead of x, y, z. Then,



in these coordinates $I = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, since

if $\vec{\omega} = \vec{\omega}_1$, $\vec{L} = I\vec{\omega}_1 = I\begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1\vec{\omega}_1$. Likewise,

if $\vec{\omega} = \omega_2$, $\vec{L} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} 0 \\ \omega_2 \\ 0 \end{bmatrix} = \lambda_2\vec{\omega}_2$.

Thus, in these coordinates, I is a diagonal matrix.

Why is this useful? In your homework you will show $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$. In principal axes, if $\vec{\omega} = \omega_1\hat{\omega}_1 + \omega_2\hat{\omega}_2 + \omega_3\hat{\omega}_3$,

$$T = \frac{1}{2}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}\cdot\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right),$$

which is easier to calculate in these coordinates. Compare this quantity to the $T = \frac{1}{2}I\omega^2$ from introductory mechanics.

So, for a rotating rigid body, how do we find the principal axes? i.e., how do we find $\vec{\omega}$ such that $\vec{L} = \lambda\vec{\omega} \Leftrightarrow I\vec{\omega} = \lambda\vec{\omega}$? This is true if $(I - \lambda 1)\vec{\omega} = 0$, where $1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity.

By the invertible matrix theorem, $(I - \lambda 1)\vec{\omega} = 0$ if and only if $\det(I - \lambda 1) = 0$. This determinant is called the <u>characteristic equation</u>. This results in a cubic equation in $\lambda \Rightarrow 3$ eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Once we find the eigenvalues, we can use $(I - \lambda_i 1)\vec{\omega} = 0$ to solve for $\vec{\omega}$.

ex) Solve for the eigenvalues/eigenvectors of the cube rotated about a corner.

$$I = \underbrace{\frac{Ma^2}{12}}_{=\mu}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} \Rightarrow I - \lambda 1 = \begin{bmatrix} 8\mu-\lambda & -3\mu & -3\mu \\ -3\mu & 8\mu-\lambda & -3\mu \\ -3\mu & -3\mu & 8\mu-\lambda \end{bmatrix}$$

Let $a = 8\mu - \lambda$, $b = -3\mu \Rightarrow I - \lambda 1 = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$

$$\Rightarrow \det(I - \lambda 1) = a(a^2 - b^2) - b(ab - b^2) + b(b^2 - ab)$$
$$= a(a^2 - b^2) + 2b(b - a) = -a(a+b)(b-a) + 2b^2(b-a)$$
$$= (b-a)[-a^2 - ab + 2b^2] = (b-a)(2b+a)(b-a)$$
$$= (2\mu - \lambda)(-11\mu + \lambda)^2 = 0 \Rightarrow \lambda_1 = 2\mu = \frac{Ma^2}{6},$$
$$\lambda_2 = \lambda_3 = 11\mu = \frac{11Ma^2}{12}$$

Note λ_1 is the same as before. What are the $\vec{\omega}$?

$$(I - \lambda_1 1)\vec{\omega}_1 = \left.\begin{bmatrix} 8\mu-\lambda & -3\mu & -3\mu \\ -3\mu & 8\mu-\lambda & -3\mu \\ -3\mu & -3\mu & 8\mu-\lambda \end{bmatrix}\right|_{\lambda=2\mu}\vec{\omega}_1 = \begin{bmatrix} 6\mu & -3\mu & -3\mu \\ -3\mu & 6\mu & -3\mu \\ -3\mu & -3\mu & 6\mu \end{bmatrix}\vec{\omega}_1 = 0$$

$$\Rightarrow \vec{\omega}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{1}{\sqrt{3}},$$ same as before!

For λ_2: $\lambda_2 = 11\mu$
$$(I - \lambda_2 1)\vec{\omega}_2 = \begin{bmatrix} -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \\ -3\mu & -3\mu & -3\mu \end{bmatrix}\vec{\omega}_2 = 0$$

Same for λ_3!

$$\Rightarrow \omega_{2,x} + \omega_{2,y} + \omega_{2,z} = 0 = \vec{\omega}_2 \cdot \vec{\omega}_1 (\sqrt{3})$$
$$\omega_{3,x} + \omega_{3,y} + \omega_{3,z} = 0 = \vec{\omega}_3 \cdot \vec{\omega}_1 (\sqrt{3})$$

$\Rightarrow \vec{\omega}_2$ and $\vec{\omega}_3$ are othogonal to $\vec{\omega}_1$, but there are many allowed choices. The eigenvectors $\vec{\omega}_2$ and $\vec{\omega}_3$ are <u>degenerate</u>, which is a direct result of the eigenvalues λ_2 and λ_3 being equal.

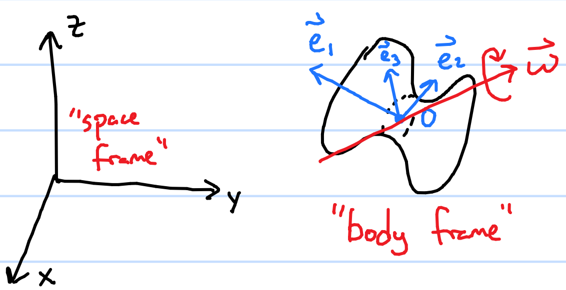
When $\vec{\omega}_1$, $\vec{\omega}_2$, and $\vec{\omega}_3$ are our basis,

$$I' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$ that is, the

inertia tensor is diagonalized in this basis.

We will use these principal axes as a basis to simplify the mathematics of the next section. However, these principal axes might be spinning in our system of interest (e.g. a twirling baton or football).

We will relate the space frame (fixed inertial observer) to the body frame (noninertial rotating principal axes) using the same methods as in chapter 9.



Assume the body is rotating at angular velocity $\vec{\omega} = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3$, the principal moments are λ_1, λ_2, and λ_3, and $\vec{L} = \lambda_1\omega_1\hat{e}_1 + \lambda_2\omega_2\hat{e}_2 + \lambda_3\omega_3\hat{e}_3$ in the body frame. Also, from intro mechanics,

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma},$$

that is, in an inertial reference frame a torque Γ induces a change in $\vec{L}$. (We specify the space frame since $\dot{\vec{L}} = \vec{\Gamma}$ is derived from N2L, hence it only holds in inertial frames) From before, we also know

$$\left(\frac{d\vec{Q}}{dt}\right)_{space} = \left(\frac{d\vec{Q}}{dt}\right)_{body} + \omega \times \vec{Q}$$

for any vector $\vec{Q}$.

Then, we can plug in $\vec{L}$ to see

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

$\Rightarrow \dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}$, where dot notation is in the body frame.

Note

$$\vec{\omega} \times \vec{L} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{vmatrix} = \begin{aligned} &\hat{e}_1(\lambda_3-\lambda_2)\omega_2\omega_3 \\ &-\hat{e}_2(\lambda_3-\lambda_1)\omega_1\omega_3 \\ &+\hat{e}_3(\lambda_2-\lambda_1)\omega_1\omega_2, \end{aligned}$$

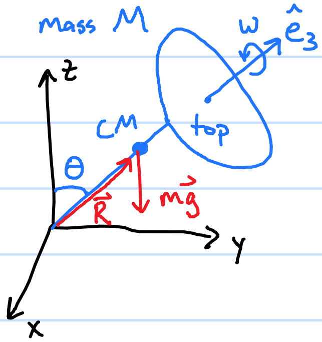
and $\dot{\vec{L}} = \lambda_1\dot{\omega}_1 + \lambda_2\dot{\omega}_2 + \lambda_3\dot{\omega}_3$. Hence,

$$\boxed{\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_1\omega_3 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned}}$$

These are Euler's equations, and are the rotational version of N2L.

In general, these equations are very unwieldy and tricky to use, since we would always need to compute the torque in terms of the basis $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ which is changing in time. Sometimes, however, problems are simple enough to be tractable.

Ex) Spinning top



A top is created by attaching a rod to a disk, as at right. Initially, assume it is spinning about its principle axis $\hat{e}_3$ with magnitude w. (Why is $\hat{e}_3$ a principal axis? You should be able to use symmetry to figure it out. Think through what the inertia tensor terms would be in the $\hat{e}_3$ direction.)

Without gravity, its angular momentum would be

$$\vec{L} = \lambda_3 \vec{w} = \lambda_3 w \hat{e}_3 .$$

No gravity $\Rightarrow$ no torque $\Rightarrow \vec{\Gamma} = 0$. Hence Euler's eqns read:

$$\lambda_1 \dot{w}_1 = (\lambda_2 - \lambda_3) w_2 w_3$$
$$\lambda_2 \dot{w}_2 = (\lambda_3 - \lambda_1) w_1 w_3$$
$$\lambda_3 \dot{w}_3 = (\lambda_1 - \lambda_2) w_1 w_2$$

Since only w_3 is nonzero, $\dot{w}_1 = \dot{w}_2 = \dot{w}_3 = 0$. Hence the object will remain spinning only about $\hat{e}_3$.

Now "turn on" gravity. This will induce a torque

$$\vec{\Gamma} = \vec{R} \times M\vec{g} = R\hat{e}_3 \times M\vec{g} = RM\, \hat{e}_3 \times \vec{g} .$$

Note $\vec{\Gamma}$ is perpendicular to $\hat{e}_3 \Rightarrow \Gamma_3 = 0$. There will be components of $\vec{\Gamma}$ in the $\hat{e}_1$ and $\hat{e}_2$ directions $\Rightarrow \Gamma_1 \neq 0,\ \Gamma_2 \neq 0$. Then, Euler's eqns are become

$$\lambda_1 \dot{w}_1 - (\lambda_2 - \lambda_3) w_2 w_3 = \Gamma_1$$
$$\lambda_2 \dot{w}_2 - (\lambda_3 - \lambda_1) w_1 w_3 = \Gamma_2$$
$$\lambda_3 \dot{w}_3 - (\lambda_1 - \lambda_2) w_1 w_2 = 0$$

By axial symmetry, $\lambda_1 = \lambda_2 \Rightarrow \dot{w}_3 = 0$ still. Hence the w_3 component remains unchanged. However, at the same time, in the space (inertial) frame,

$$\dot{\vec{L}} = \vec{\Gamma} \Rightarrow \lambda_3 \dot{\vec{w}}_3 = \vec{R} \times M\vec{g}$$
$$\Rightarrow \lambda_3 w_3 \dot{\hat{e}}_3 = RMg\ \hat{z} \times \hat{e}_3$$
$$\Rightarrow \dot{\hat{e}}_3 = \frac{RMg}{\lambda_3 w_3} \hat{z} \times \hat{e}_3 \equiv \vec{\Omega} \times \hat{e}_3$$

Hence, the direction of $\hat{e}_3$ is changing, while its magnitude remains the same. This is true initially (when $w_1 = w_2 = 0$), but what happens when w_1 and w_2 become nonzero due to the torque Γ_1 and Γ_2? In the limit Γ_1 and Γ_2 are small, if w_3 is sufficiently rapid, then $\hat{e}_1$ and $\hat{e}_2$ are rotating rapidly in the space frame, and hence Γ_1 and Γ_2 are oscillating rapidly in the body frame (since e.g. Γ_1 is the component of $\vec{\Gamma}$ in the $\hat{e}_1$ direction). Hence we may neglect the contributions of w_1 and w_2 in the weak torque limit.

Next let's consider the special case $\vec{\Gamma} = 0$ (i.e. no forces acting on the body), with $\lambda_1 \neq \lambda_2 \neq \lambda_3$.

Further assume our body is initialized w/ axis of rotation $\vec{\omega} = (0, 0, \omega_3)$, then gently kicked so that $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ with $\omega_1, \omega_2 \ll \omega_3$.

Then $\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2$, so if ω_1 and ω_3 are small, then $\dot{\omega}_3$ is $(\text{small})^2$ ⇒ ω_3 is approx constant

Then:
$$\lambda_1 \dot{\omega}_1 = [(\lambda_2 - \lambda_3)\omega_3]\omega_2$$
$$\lambda_2 \dot{\omega}_2 = [(\lambda_3 - \lambda_1)\omega_3]\omega_1$$
Recall $\dot{\omega}_3 = 0$, and note
$$\lambda_1 \ddot{\omega}_1 = [(\lambda_2 - \lambda_3)\omega_3]\dot{\omega}_2 = \frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)\omega_3^2}{\lambda_1 \lambda_2}\omega_1$$
$$\Rightarrow \ddot{\omega}_1 = -\left[\frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}\omega_3^2\right]\omega_1 \Rightarrow \ddot{\omega}_1 = -k\,\omega_1$$

Thus, if $k = (\lambda_3-\lambda_2)(\lambda_3-\lambda_1)\omega_3^2/(\lambda_1\lambda_2) > 0$, then
$\omega_1 \sim A\cos(\sqrt{k}\,t) + B\sin(\sqrt{k}\,t)$ ⇒ stable oscillations.

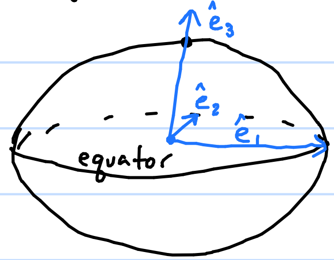
If $k < 0$, then $\ddot{\omega}_1 = \underbrace{-k_1}_{>0}\omega_1 \Rightarrow \omega_1 = A e^{\sqrt{-k_1}\,t} + B e^{-\sqrt{-k_1}\,t}$
⇒ exponential growth

Therefore, the geometry of an object determines whether its rotations are stable about a principal axis.

Next let's consider the special case $\vec{\Gamma} = 0$ (i.e. no forces acting on the body), with $\lambda_1 = \lambda_2 \neq \lambda_3$, so $\dot{\omega}_3 = 0$ by the 3rd Euler eqn.

A "squished sphere" (ellipsoid w/ equal minor axes) satisfies this:

(e.g. the Earth's bulging equator leads to $\lambda_1 = \lambda_2 \neq \lambda_3$.)



Here, $k = (\lambda_3 - \lambda_1)^2 \omega_3^2 / \lambda_1^2 > 0$ necessarily, so the body will oscillate if initialized with $\vec{\omega} = \omega\hat{e}_3$, then kicked so that $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ w/ ω_1, ω_2 small.

In this case we can solve for ω_1 and ω_2 as a fcn of time:
$$\dot{\omega}_1 = \frac{(\lambda_1-\lambda_3)\omega_3}{\lambda_1}\omega_2 \equiv \Omega_b\,\omega_2$$
$$\dot{\omega}_2 = -\frac{(\lambda_1-\lambda_3)\omega_3}{\lambda_1}\omega_1 \equiv -\Omega_b\,\omega_1$$

This coupled ODE can be solved w/ the same trick $\eta = \omega_1 + i\omega_2$ as before, to find

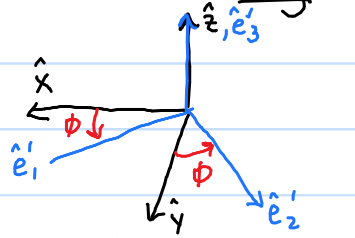
$\omega_1(t) = \omega_0 \cos(\Omega_b t)$
$\omega_2(t) = -\omega_0 \sin(\Omega_b t)$ ⇒ $\vec{L} = \lambda_1 \omega_1(t)\,\hat{e}_1 + \lambda_1 \omega_2(t)\,\hat{e}_2 + \lambda_3 \omega_3\,\hat{e}_3$
$\omega_3(t) = \omega_3$

This says, if rotation is not initialized solely in the $\hat{e}_3$ direction, then the angular momentum will precess in the $\hat{e}_1$ and $\hat{e}_2$ directions. This is called free precession.

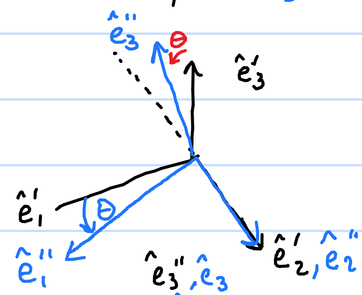
We formulated Euler's equations in the <u>body frame</u>, but it is hard to keep track of where $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ are pointing in the <u>space frame</u>. How do we relate the two? With <u>Euler angles</u>.

Idea: Begin with $\overbrace{\hat{e}_1, \hat{e}_2, \hat{e}_3}^{\text{body}}$ aligned w/ $\overbrace{\hat{x}, \hat{y}, \hat{z}}^{\text{space}}$. 3 angles (θ, ϕ, ψ) allow us to describe <u>any</u> body frame orientation.

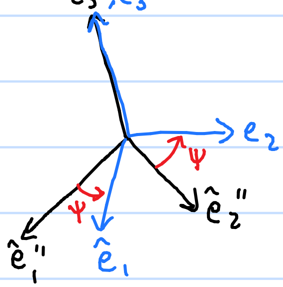
a) Rotate about $\hat{z}$ by an amount ϕ



b) Rotate about $\hat{e}_2'$ by an amount θ



c) Rotate about $\hat{e}_3'' = \hat{e}_3$ by an amount ψ



While these diagrams may look confusing, the main point is simple: the orientation of any rigid body may be <u>encoded</u> in 3 angles, that describe how to reach the desired orientation from the default coordinates $\hat{x}, \hat{y}, \hat{z}$.

How can we relate these Euler angles to the body's <u>actual</u> angular velocity? Well, we can decompose the problem:

a) If ϕ varies, then the $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ frame will rotate w/ angular velocity $\vec{\omega}_a = \dot{\phi}\, e_3' = \dot{\phi}\, \hat{z}$

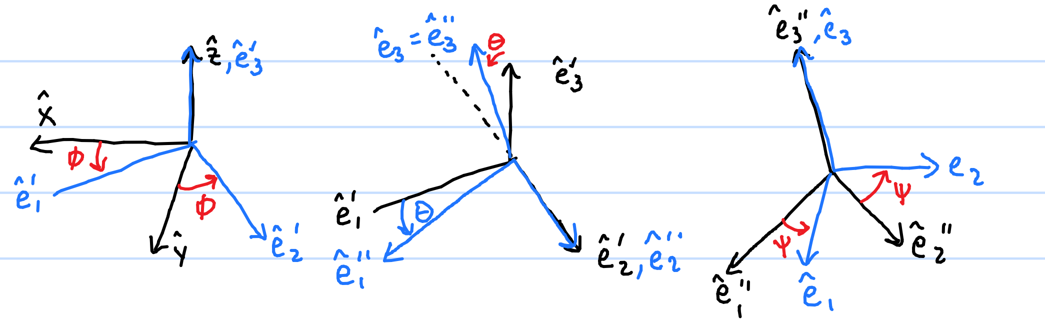
b) If θ varies, the e_1'', e_2'', e_3'' frame will rotate w/ ang. vel. $\vec{\omega}_b = \dot{\theta}\, \hat{e}_2'' = \dot{\theta}\, \hat{e}_2'$

c) If ψ varies the $\hat{e}_1, \hat{e}_2, \hat{e}_3$ frame will rotate w/ ang. vel. $\vec{\omega}_c = \dot{\psi}\, \hat{e}_3$

Thus we can think of varying ϕ, θ, ψ as corresponding to a rotation $\vec{\omega} = \dot{\phi}\, \hat{z} + \dot{\theta}\, \hat{e}_2' + \dot{\psi}\, \hat{e}_3$.

We want to describe $\vec{\omega}$ in a basis of principal axes so that we can use $\vec{L} = \sum_i \lambda_i \vec{\omega}_i$, but right now we have $\vec{\omega}$ in terms of $\hat{z}, \hat{e}_2', \hat{e}_3$.

If we assume $\lambda_1 = \lambda_2$, then there are lots of $\hat{e}_1$ and $\hat{e}_2$ that are principal axes, e.g. $\hat{e}_1''$ and $\hat{e}_2''$ from the middle figure. It will be easier to compute $\vec{\omega}$ if we use these axes.



So, if $\lambda_1 = \lambda_2$ and we use $\hat{e}_1'', \hat{e}_2'', \hat{e}_3$ as our principal axes, then note $\hat{z} = (\cos\theta)\hat{e}_3 - (\sin\theta)\hat{e}_1''$ — here, check $\theta = 0 \Rightarrow \hat{z} = \hat{e}_3$, and $\theta = \frac{\pi}{2} \Rightarrow \hat{z} = -\hat{e}_1''$.

Then, we may rewrite $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3$ as

$$\vec{\omega} = -\dot{\phi}\sin\theta\,\hat{e}_1'' + \dot{\theta}\hat{e}_2'' + \underline{(\dot{\psi} + \dot{\phi}\cos\theta)}\hat{e}_3$$

Thus, when $\lambda_1 = \lambda_2$ we have written $\vec{\omega}$ in terms of its principal axes, so

$$\vec{L} = -\lambda_1\dot{\phi}_1\sin\theta\,\hat{e}_1'' + \lambda_1\dot{\theta}\hat{e}_2'' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3,$$

and since kinetic energy is $T = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$,

$$T = \frac{1}{2}\lambda_1\left[\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right] + \frac{1}{2}\lambda_3\left[\dot{\psi} + \dot{\phi}\cos\theta\right]^2$$

Note this is <u>independent</u> of our choice of $\hat{e}_1'', \hat{e}_2'', \hat{e}_3$.

Now we can do Lagrangian mechanics for a spinning top (with $\lambda_1 = \lambda_2$). The potential energy is $U = mgR\cos\theta$. Then,

$$\mathcal{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - M_gR\cos\theta$$

$q_i = \theta, \phi, \psi$. Notice ϕ and ψ are <u>ignorable</u>.

The E-L eqns are:

θ eqn: $\lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2\sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + M_gR\sin\theta$

ϕ eqn: $p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \underline{\lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta} = \text{const}$

ψ eqn: $p_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \underbrace{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}_{=L_3} = \text{const}$

Notice p_ψ is simply the angular momentum in the $\hat{e}_3$ direction, which is conserved.

Also, we can use $\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1''$ to see

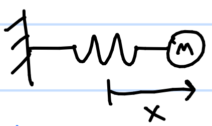
$$L_z = \underbrace{\lambda_3\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta) + \lambda_1\dot{\phi}\sin^2\theta}_{=L_z}$$

Notice p_ϕ is simply the angular momentum in the z-direction, which is conserved since there is no torque in the z-direction. (We saw this in Euler's equations, too.)

Therefore we may fully understand rigid body rotation in terms of Lagrangian mechanics and conserved quantities.

Chapter 11: Coupled oscillators + normal modes

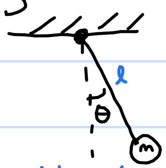
The harmonic oscillator is ubiquitous in physics, since it is mathematically simple, and a useful approximation to any system relaxing to a nearby equilibria:



harmonic oscillator

$(F = m\ddot{x} = -kx)$

$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$



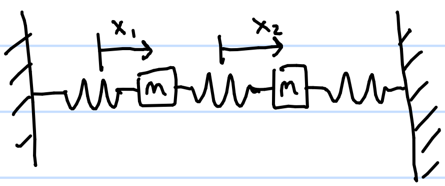
pendulum (at small θ, $\ddot{\theta} = -\frac{g}{L}\theta$)

$\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mg(1-\theta^2)$

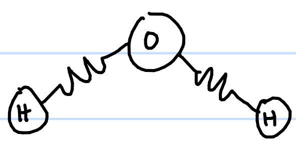
ball on a hill

$\mathcal{L} = \frac{1}{2}(\dot{x}^2+\dot{y}^2) - mgax^2$

Often, these harmonic oscillators are coupled (or, rather, often we can approximate physical systems as coupled harmonic oscillators):

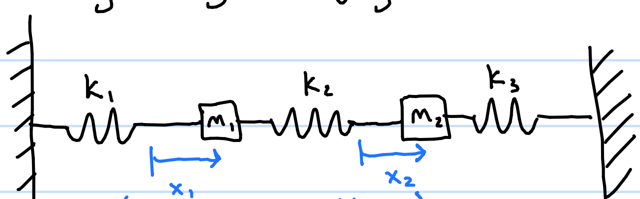


coupled harmonic oscillator



water molecule (chemical bonds act like springs!)

Let's begin by studying a two-mass coupled HO:



(distances from equilibrium)

We already know how to describe this system (both N2L and Lagrange are simple):

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$U = \frac{1}{2}k_1x_1^2 + \underbrace{\frac{1}{2}k_2(x_2-x_1)^2}_{\text{energy in spring 2 fcn of both } x_1 \text{ and } x_2} + \frac{1}{2}k_3x_2^2$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2-x_1)^2 - \frac{1}{2}k_3x_2^2$$

$$\overset{\text{E-L}}{\Rightarrow} \quad m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2-x_1) = x_1(-k_1-k_2) + x_2(k_2)$$

$$m_2\ddot{x}_2 = -k_2(x_2-x_1) - k_3x_2 = x_1(k_2) + x_2(-k_2-k_3)$$

$$\Rightarrow M\ddot{\vec{x}} = -K\vec{x}, \quad \text{with} \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}, \quad \text{and} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We are familiar enough with this problem to expect sinusoidal solutions. In particular, consider a "ansatz" solution

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underset{\substack{\uparrow \\ \text{real part}}}{\text{Re}}\,\vec{z} = \text{Re}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \text{Re}\begin{bmatrix} a_1 e^{i\omega t} \\ a_2 e^{i\omega t} \end{bmatrix} = \begin{matrix} A_1\cos(\omega t - \delta_1) \\ A_2\cos(\omega t - \delta_2) \end{matrix},$$

(a_1, a_2: complex #)

for some complex vector exponential $\vec{z}$. Note that if $\vec{z}(t) = \vec{x}(t) + i\vec{y}(t)$ satisfies $M\ddot{\vec{z}} = -K\vec{z}$, then $M(\ddot{\vec{x}} + i\ddot{\vec{y}}) = -K(\vec{x} + i\vec{y}) \Rightarrow$ both $\vec{x}$ and $\vec{y}$ satisfy $M\ddot{\vec{x}} = -K\vec{x}$

Thus, we may consider the complex soln $z(t)$ and at the end take its real part.

Therefore, consider $\vec{z}(t) = \begin{bmatrix} a_1 e^{i\omega t} \\ a_2 e^{i\omega t} \end{bmatrix}$ with a_1 and a_2 complex. Plugging $\vec{z}$ in,

$$M\ddot{\vec{z}} = -K\vec{z} \Rightarrow -M\omega^2\vec{z} = -K\vec{z} \Rightarrow (K-\omega^2 M)\vec{z} = 0.$$

Thus, for our ansatz $\vec{z}$ to work, either $\vec{z} = 0$ (trivial soln is boring) or $K\vec{z} = \omega^2 M\vec{z}$.

Note: this is very similar to an eigenvalue problem. In fact, define $A = \begin{bmatrix} (k_1+k_2)/m_1 & -k_2/m_1 \\ -k_2/m_2 & (k_2+k_3)/m_2 \end{bmatrix}$ and this eqn becomes

$$\underbrace{A}_{\text{matrix}}\vec{z} = \underbrace{\omega^2}_{\text{real \#, eigenvalue}}\overbrace{\vec{z}}^{\text{eigenvector}}$$

Therefore, choosing the proper coefficients for our ansatz requires solving an eigenvalue equation! When does $(K-\omega^2 M)\vec{z} = 0$? When $\det(K-\omega^2 M) = 0$. (This is the invertible matrix theorem, aka the fundamental theorem of linear algebra.)

$$\det(K-\omega^2 M) = \begin{vmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2+k_3-\omega^2 m_2 \end{vmatrix} = 0$$

$$\Rightarrow (k_1+k_2-\omega^2 m_1)(k_2+k_3-\omega^2 m_2) - k_2^2 = 0$$

$\Rightarrow$ quadratic eqn for $\omega^2 \Rightarrow$ two solns ω_1 and ω_2 called normal frequencies

$\Rightarrow$ Solns will look like $z_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_1 t}$ and $z_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{i\omega_2 t}$. Different frequencies of oscillation!

ex) Special case: $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$.

In this case, $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ and $K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$. Then

$$K - \omega^2 M = \begin{bmatrix} 2k-\omega^2 m & -k \\ -k & 2k-\omega^2 m \end{bmatrix},$$ and

$$\det(K-\omega^2 M) = (2k-\omega^2 m)^2 - k^2$$ use $a^2 - b^2 = (a-b)(a+b)$

$$= (k-\omega^2 m)(3k-\omega^2 m) = 0$$

$$\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Now that we have found the normal frequencies ω_1 and ω_2, we must find the normal modes $\vec{a}$ coefficients of sinusoidal motion

(normal frequencies $\Leftrightarrow$ eigenvalues, normal modes $\Leftrightarrow$ eigenvectors)

The first normal mode has frequency $\omega_1 = \sqrt{k/m}$.

We need $(K-\omega_1^2 M)\vec{z} = \vec{0} \Rightarrow (K-\omega_1^2 M)\vec{a}e^{i\omega_1 t} = \vec{0}$

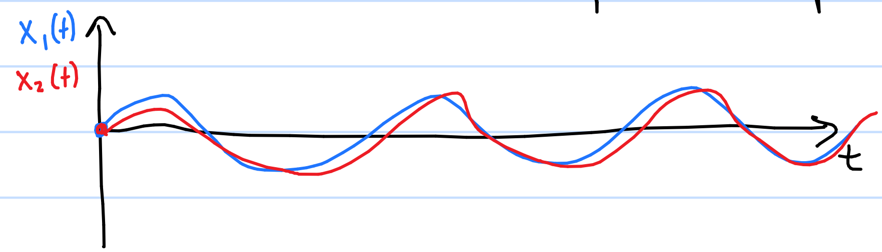
$\Rightarrow (K-\omega_1^2 M)\vec{a} = \vec{0}$.

$$(K-\omega_1^2 M)\vec{a} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{0} \Rightarrow \begin{matrix} k(a_1-a_2) = 0 \\ -k(a_1-a_2) = 0 \end{matrix}$$

$$\Rightarrow a_1 = a_2 = Ae^{-i\delta} \Rightarrow \vec{z}(t) = \begin{bmatrix} Ae^{-i\delta} \\ Ae^{-i\delta} \end{bmatrix} e^{i\omega_1 t}. \quad \vec{x}(t) = \mathrm{Re}(\vec{z}(t))$$

$$\Rightarrow x_1(t) = x_2(t) = A\cos(\omega_1 t - \delta)$$

$\Rightarrow$ the two masses oscillate in phase w/ amplitude A.



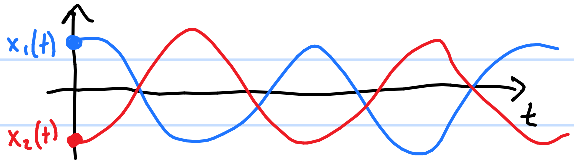
The second normal mode has frequency $\omega_2 = \sqrt{3k/m}$

$$\Rightarrow (K-\omega_2^2 M)\vec{a} = \begin{bmatrix}-k & -k\\ -k & -k\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} \Rightarrow -k(a_1+a_2) = 0$$

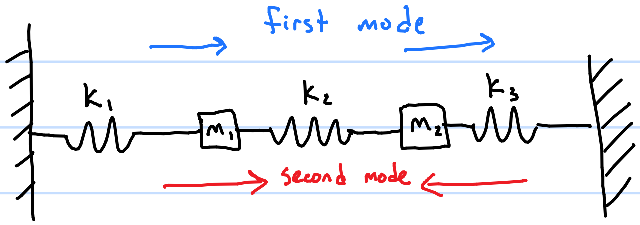
$$\Rightarrow a_1 = -a_2 = Ae^{-i\delta} \Rightarrow \vec{z}(t) = \begin{bmatrix}Ae^{-i\delta}\\ -Ae^{-i\delta}\end{bmatrix} e^{i\omega_2 t}, \quad \vec{x}(t) = \mathrm{Re}[\vec{z}]$$

$$\Rightarrow x_1(t) = A\cos(\omega_2 t - \delta)$$
$$x_2(t) = -A\cos(\omega_2 t - \delta)$$

$\Rightarrow$ two masses oscillating exactly out of phase



Thus, the two normal modes are:



The general soln is

$$x_1(t) = a_1\cos(\omega_1 t - \delta_1) + b_1\cos(\omega_2 t - \delta_2)$$
$$x_2(t) = a_1\cos(\omega t - \delta_1) - b_1\cos(\omega_2 t - \delta_2)$$

Recall, in chapter 10 we were faced with an ugly matrix equation $\begin{bmatrix}L_x\\ L_y\\ L_z\end{bmatrix} = \begin{bmatrix}I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$, which we were able to simplify to $\vec{L} = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\vec{\omega}$ by cleverly choosing new coordinates. We can do the same thing here!

Let's choose new coordinates according to our eigenvectors: $\xi_1 = \frac{1}{2}(x_1 + x_2)$, $\xi_2 = \frac{1}{2}(x_1 - x_2)$. Then $\ddot{\xi}_1 = \frac{1}{2}(\ddot{x}_1 + \ddot{x}_2)$, and $\ddot{\xi}_2 = \frac{1}{2}(\ddot{x}_1 - \ddot{x}_2)$. $m\ddot{x}_1 = -2kx_1 + kx_2$
$m\ddot{x}_2 = kx_1 - 2kx_2$

$$\Rightarrow m\ddot{\xi}_1 = \tfrac{1}{2}(-kx_1 - kx_2) = -k\xi_1$$
$$m\ddot{\xi}_2 = \tfrac{1}{2}(-3kx_1 + 3kx_2) = -3k\xi_2$$

$$\left(M' = \begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}, \; K' = \begin{bmatrix}k & 0\\ 0 & 3k\end{bmatrix}\right)$$

These are decoupled ODE's! They can be solved immediately:

$$\xi_1(t) = A_1\cos\left(\sqrt{k/m}\, t - \delta_1\right) = x_1(t) + x_2(t)$$
$$\xi_2(t) = A_2\cos\left(\sqrt{3k/m}\, t - \delta_2\right) = x_1(t) - x_2(t)$$

Rewriting the ODE in this form is known as the normal form decomposition:

Each $\vec{a}_{(i)}$ is a N-dim vector

Consider N coupled oscillators described by gen'l coords $q_1, \ldots, q_N$. Then $\vec{q}(t) = \sum_{i=1}^{N} \vec{a}_{(i)}\cos(\omega_i t - \delta_i)$, i.e. its soln may be written as the sum of N normal modes w/ normal frequencies ω_i. Each of these $\vec{a}_{(i)}$ and ω_i must satisfy the eigenvalue eqn $(K - \omega_i^2 M)\vec{a}_{(i)} = 0$. Calling each normal mode $\xi_i(t)$, $\vec{q}(t) = \sum_i \xi_i(t)\vec{a}_{(i)}$. Also, $\vec{q}$ obeys the system's eqn of motion $M\ddot{\vec{q}} = -K\vec{q}$. Then

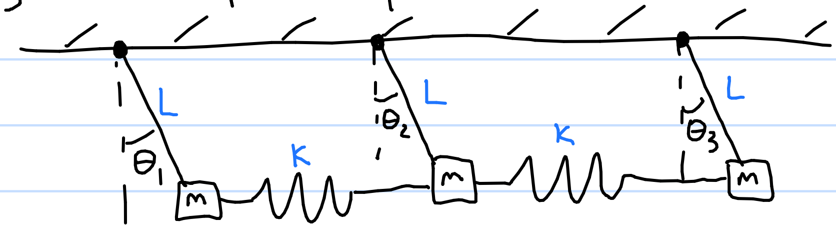
$$\sum_i \ddot{\xi}_i(t) M\vec{a}_{(i)} = -\sum_i \xi_i(t) K\vec{a}_{(i)} = -\sum_i \xi_i(t)\,\omega_i^2 M\vec{a}_{(i)}.$$

$\Rightarrow \sum_i \left[\ddot{\xi}_i(t) + \xi_i(t)\,\omega_i^2\right] M\vec{a}_{(i)} = 0$. This must be true for any normal mode, i.e. I can choose $\vec{a}_{(i)} = 1$ arbitrarily. Hence

$$\ddot{\xi}_i(t) = -\omega_i^2\,\xi_i(t) \text{ for each } i \Rightarrow \xi_i = A_i\cos(\omega_i t - \delta_i)$$

Therefore, we can decouple any coupled harmonic oscillator into its normal modes!

Lastly, we will extend our approach to study 3 coupled pendulums:



Consider $\theta_1, \theta_2, \theta_3$ small. Then the Lagrangian is:

$$\mathcal{L} = \overbrace{\frac{1}{2} mL^2\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2\right)}^{T} + \overbrace{mgL\left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)}^{U_{grav}}$$

U_{spring}

$$- \frac{1}{2} k\left[(L\sin\theta_2 - L\sin\theta_1)^2 + (L\sin\theta_3 - L\sin\theta_2)^2\right]$$

$$\approx \frac{1}{2} mL^2\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2\right) + 3mgL - \frac{mgL}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right)$$

$$- \frac{1}{2} kL^2\left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2\right]$$

$$\Rightarrow mL^2\ddot\theta_1 = -mgL\theta_1 + kL^2(\theta_2 - \theta_1)$$

$$mL^2\ddot\theta_2 = -mgL\theta_2 - kL^2(\theta_2 - \theta_1) + kL^2(\theta_3 - \theta_2)$$

$$mL^2\ddot\theta_3 = -mgL\theta_3 - kL^2(\theta_3 - \theta_2)$$

$$\Rightarrow \begin{bmatrix} mL^2 & 0 & 0 \\ 0 & mL^2 & 0 \\ 0 & 0 & mL^2 \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{bmatrix} = -\begin{bmatrix} mgL + kL^2 & -kL^2 & 0 \\ -kL^2 & mgL + kL^2 + kL^2 & -kL^2 \\ 0 & -kL^2 & mgL + kL^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\Leftrightarrow M\ddot{\vec\theta} = -K\vec\theta.$$

Guess $\vec\theta(t) = \mathrm{Re}[\vec z(t)] = \mathrm{Re}[\vec a e^{i\omega t}]$

$$\Rightarrow -\omega^2 M\vec z = -K\vec z \Rightarrow (K - \omega^2 M)\vec z = 0$$

$$\Rightarrow (K - \omega^2 M)\vec a = 0 \Rightarrow \text{Ansatz}$$

works when $\det(K - \omega^2 M) = 0$.

$$K - \omega^2 M = \begin{bmatrix} mgL + kL^2 - \omega^2 mL^2 & -kL^2 & 0 \\ -kL^2 & mgL + 2kL^2 - \omega^2 mL^2 & -kL^2 \\ 0 & -kL^2 & mgL + kL^2 - \omega^2 mL^2 \end{bmatrix}$$

$$\det(K - \omega^2 M) = (mgL + kL^2 - \omega^2 mL^2)^2(mgL + 2kL^2 - \omega^2 mL^2)$$

$$- (mgL + kL^2 - \omega^2 mL^2)(kL^2)^2$$

$$+ kL^2(-kL^2)(mgL + kL^2 - \omega^2 mL^2)$$

Let $a = mgL + kL^2 - \omega^2 mL^2$, $b = kL^2$

$$\Rightarrow \det(K - \omega^2 M) = a^2(a + b) - ab^2 - b^2 a$$

$$= a[a^2 + ab - 2b^2] = a(a + 2b)(a - b)$$

$$0 = (\underbrace{mgL + kL^2 - \omega^2 mL^2}_{\omega_1})(\underbrace{mgL + 3kL^2 - \omega^2 mL^2}_{\omega_2})(\underbrace{mgL - \omega^2 mL^2}_{\omega_0})$$

$$\Rightarrow \omega_0 = \sqrt{g/L}$$

$$\omega_1 = \sqrt{\frac{g}{L} + \frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{g}{L} + 3\frac{k}{m}}$$

What are the eigenmodes?

$$[K^2 - \omega^2 M]_{\omega = \omega_0} = \begin{bmatrix} kL^2 & -kL^2 & 0 \\ -kL^2 & 2kL^2 & -kL^2 \\ 0 & -kL^2 & kL^2 \end{bmatrix}$$

Need $(K^2 - \omega_0^2 M)\vec a_{(1)} = 0 \Rightarrow \vec a_{(1)} = \begin{bmatrix} A \\ A \\ A \end{bmatrix} e^{-i\delta_1}$

$$\Rightarrow \theta_1(t) = \theta_2(t) = \theta_3(t) = A\cos(\omega_0 t - \delta_1)$$

in phase!

$$K-\omega^2 M = \begin{bmatrix} mgL+kL^2-\omega^2 mL^2 & -kL^2 & 0 \\ -kL^2 & mgL+2kL^2-\omega^2 mL^2 & -kL^2 \\ 0 & -kL^2 & mgL+kL^2-\omega^2 mL^2 \end{bmatrix}$$

Next, $\omega_1 = \sqrt{\frac{g}{L}+\frac{k}{m}}$

$$\Rightarrow K-\omega_1^2 M = \begin{bmatrix} 0 & -kL^2 & 0 \\ -kL^2 & kL^2 & -kL^2 \\ 0 & -kL^2 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{a}_{(2)} = \begin{bmatrix} A \\ 0 \\ -A \end{bmatrix} e^{-i\delta_2} \Rightarrow \begin{aligned} \theta_1(t) &= A\cos(\omega_1 t - \delta_1) \\ \theta_2(t) &= 0 \\ \theta_3(t) &= -A\cos(\omega_1 t - \delta_1) \end{aligned}$$

Lastly, $\omega_2 = \sqrt{\frac{g}{L}+\frac{3k}{m}}$

$$K-\omega_2^2 M = \begin{bmatrix} -2kL^2 & -kL^2 & 0 \\ -kL^2 & -kL^2 & -kL^2 \\ 0 & -kL^2 & -2kL^2 \end{bmatrix}$$

$$[K-\omega_2^2 M]\,\vec{a}_{(3)} = 0$$

$$\Rightarrow \vec{a}_{(3)} = \begin{bmatrix} A \\ -2A \\ A \end{bmatrix} e^{-i\delta_3}$$

$$\Rightarrow \begin{aligned} \theta_1(t) &= A\cos(\omega_2 t - \delta_3) \\ \theta_2(t) &= -2A\cos(\omega_2 t - \delta_3) \\ \theta_3(t) &= A\cos(\omega_2 t - \delta_3) \end{aligned}$$

In general, θ_1, θ_2, and θ_3 may be any linear combination of these solns

(i.e. may have components of
$a\cos(\omega_0 t) + b\cos(\omega_1 t) + c\cos(\omega_2 t)$)

Physically, these modes mean:

ω_0: $\rightarrow$ $\rightarrow$ $\rightarrow$

ω_1: $\rightarrow$ 0 $\leftarrow$

ω_2: $\rightarrow$ $\longleftarrow$ $\rightarrow$

Chapter 13: Hamiltonian Mechanics

Recall Lagrangian mechanics:

$$\mathcal{L} = T - U, \quad \mathcal{L} = \mathcal{L}(q_1, \dots, q_N, \dot{q}_1, \dots, \dot{q}_N, t)$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right); \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ ... generalized momenta}$$

Let $\vec{q} = (q_1, \dots, q_N)$ and $\vec{p} = (p_1, \dots, p_N) \Rightarrow \mathcal{L} = \mathcal{L}(\vec{q}, \vec{p}, t)$

In Hamiltonian mechanics we trade $\mathcal{L}(\vec{q}, \vec{p}, t)$ for a new function $\mathcal{H} = \mathcal{H}(\vec{q}, \vec{p}, t)$:

(1) $\mathcal{H}(\vec{q}, \vec{p}, t)$ ... Hamiltonian

(2) $(\vec{q}, \vec{p})$ ... phase space that forms basis for statistical mechanics

(3) Will derive $2N$ 1st order ODEs from $\mathcal{H}$. (These will replace the N 2nd order Lagrange eqns)

(4) Hamiltonian mechanics leads slowly into the standard formulation of quantum mechanics

Define the Hamiltonian $\mathcal{H}$ as $\mathcal{H} \equiv \sum_i p_i \dot{q}_i - \mathcal{L}$.

In particular, we require $\mathcal{H} = \mathcal{H}(\vec{q}, \vec{p}, t)$ and <u>not</u> $\mathcal{H}(\vec{q}, \dot{\vec{q}}, t)$. Thus, we will need to <u>invert</u> $\dot{q}_i$, i.e. find $\dot{q}_i = \dot{q}_i(\vec{q}, \vec{p}, t)$, and substitute this quantity everywhere there is a $\dot{q}_i$.

$\dot{q}_i = \dot{q}_i(\vec{q}, \vec{p}, t)$

Then, $$\mathcal{H} = \sum_i p_i \boxed{\dot{q}_i} - \mathcal{L}(\vec{q}, \boxed{\dot{\vec{q}}}, t) = \mathcal{H}(\vec{q}, \vec{p}, t)$$

Consider the quantity

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_j \Big(\underbrace{\cancel{\frac{\partial p_j}{\partial q_i}}}_{0 \text{ since } \vec{q} \text{ and } \vec{p} \text{ are independent vars!}} \dot{q}_j(\vec{q}, \vec{p}, t) + p_j \frac{\partial \dot{q}_j(\vec{q}, \vec{p}, t)}{\partial q_i} \Big)$$
$$\underbrace{- \frac{\partial \mathcal{L}}{\partial q_i} - \sum_j \underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}}_{= p_j} \frac{\partial \dot{q}_j}{\partial q_i}}_{\text{chain rule since } \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)}$$

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial q_i} = \sum_j \left[p_j \frac{\partial \dot{q}_j}{\partial q_i} - p_j \frac{\partial \dot{q}_j}{\partial q_i} \right] - \overbrace{\frac{\partial \mathcal{L}}{\partial q_i}}^{= \frac{d}{dt} p_i \text{ by E-L}} = -\dot{p}_i$$

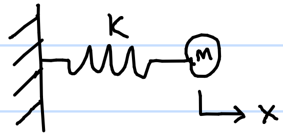
Similarly,

$$\frac{\partial \mathcal{H}}{\partial p_i} = \overbrace{\sum_j \frac{\partial p_j}{\partial p_i} \dot{q}_j}^{= \dot{q}_i(\vec{q}, \vec{p}, t)} + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i}$$
$$- \sum_j \left[\frac{\partial \mathcal{L}}{\partial q_j} \underbrace{\cancel{\frac{\partial q_j}{\partial p_i}}}_{0} + \underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}}_{p_j} \frac{\partial \dot{q}_j}{\partial p_i} \right] = \dot{q}_i$$

Thus,
$$\boxed{\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i, \quad \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i}$$ Hamilton's Equations

These equations provide 1st order equations of motion for our phase space variables $\vec{q}$ and $\vec{p}$. They replace (and are equivalent to) Lagrange's eqns.

ex) 1-D spring-mass system



Step 1: Compute Lagrangian

$$T = \tfrac{1}{2} m \dot{x}^2 , \quad U = \tfrac{1}{2} k x^2 = \frac{m\omega^2}{2} x^2$$

$\omega = \sqrt{k/m}$

$$\Rightarrow \mathcal{L} = \tfrac{1}{2} m \left[\dot{x}^2 - \omega^2 x^2 \right]$$

Step 2: Find conjugate momenta, and invert to find $\dot{q} = \dot{q}(q, p)$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \quad \Rightarrow \dot{x} = \frac{p}{m}$$

$$\Rightarrow \mathcal{L}(x,p) = \tfrac{1}{2} m \left[\left(\frac{p}{m}\right)^2 - \omega^2 x^2 \right]$$

Step 3: Compute Hamiltonian $\mathcal{H} = p\dot{q} - \mathcal{L}$

$$\mathcal{H} = \underbrace{p\dot{q}}_{= p^2/m} - \left[\frac{p^2}{2m} - \tfrac{1}{2} m\omega^2 x^2 \right]$$

$$\Rightarrow \boxed{\mathcal{H} = \underbrace{\frac{p^2}{2m}}_{T} + \underbrace{\tfrac{1}{2} m\omega^2 x^2}_{U}}$$

Step 4: Compute Hamilton's Eqns:

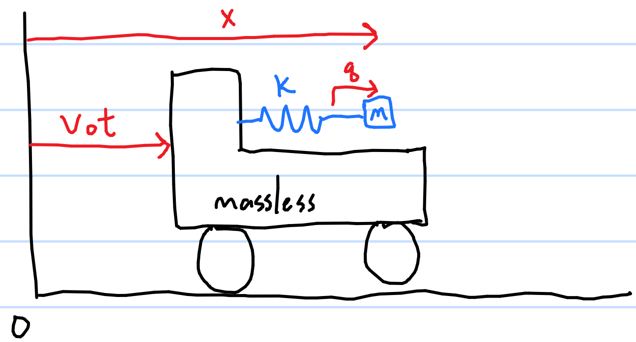
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} , \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -m\omega^2 x = -kx$$

(can always combine into $m\ddot{x} = -kx$)

Note that in this case, $\mathcal{H}$ is just the energy of the system (which is constant since $\frac{\partial \mathcal{L}}{\partial t} = 0$).

In general, for the Hamiltonian to be the energy of the system, we need our position coords $\vec{r}_i$ to only be functions of our gen'l coords q_i, i.e. $\vec{r}_i = \vec{r}_i(q_1, \dots, q_N)$. When this is the case q_i are called "natural coords"

ex) Unnatural coords with $\mathcal{H} \neq E_{tot}$



Choose the unnatural coord q. Note $\vec{r} = \vec{r}(t) = (v_0 t + q)\hat{x}$

$$\Rightarrow T = \frac{m}{2}(\dot{q} + v_0)^2 , \quad U = \tfrac{1}{2} k q^2 ,$$

$$\mathcal{L} = \frac{m}{2}(\dot{q} + v_0)^2 - \tfrac{1}{2} k q^2 \quad \Rightarrow p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m(\dot{q} + v_0)$$

$$\Rightarrow \dot{q} = \frac{p}{m} - v_0 \quad \Rightarrow T = p^2/2m , \quad \text{and}$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = p\left(\frac{p}{m} - v_0\right) - p^2/2m + \tfrac{1}{2} k q^2$$

$$= \frac{p^2}{2m} - p v_0 + \tfrac{1}{2} k q^2 = T + U - p v_0 \neq E$$

So, though $\mathcal{H}$ is constant, since $\vec{r} = \vec{r}(q, t)$ our coordinate q is <u>not</u> natural and thus $\mathcal{H} \neq E$.

Thus, whenever possible we shall try to use natural coords so that $\mathcal{H} = E_{tot}$.

Next consider the case $\mathcal{H} = \mathcal{H}(\vec{q}, \vec{p})$ so that $\mathcal{H}$ is not an explicit fcn of time. Then:

$$\frac{d}{dt}\mathcal{H} = \sum_i \left[\underbrace{\frac{\partial \mathcal{H}}{\partial q_i}}_{=-\dot{p}_i} \dot{q}_i + \underbrace{\frac{\partial \mathcal{H}}{\partial p_i}}_{=\dot{q}_i} \dot{p}_i \right] + \cancelto{0}{\frac{\partial \mathcal{H}}{\partial t}} \quad \text{by assumption}$$

$\Rightarrow \frac{d}{dt}\mathcal{H} = 0$. Thus if $\mathcal{H}$ is not an explicit function of time, then $\mathcal{H}$ is conserved.

Then, it will be most convenient to consider systems with $\frac{\partial \mathcal{H}}{\partial t} = 0$ in their natural coords $(\Rightarrow \mathcal{H} = E_{tot} = \text{const})$

Generalized Form of Hamilton's Eqns

Hamiltonian mechanics allows us to track the evolution of any observable quantity, i.e. $A = A(\vec{q}, \vec{p}, t)$. "A" could be position, or momentum, or temperature, or pressure, or probability density. As long as we can write it as a fcn of $\vec{q}$ and $\vec{p}$:

$$\frac{dA}{dt} = \sum_i \left[\frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right] + \frac{\partial A}{\partial t}$$

$$= \underbrace{\sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} \right]}_{\equiv \{A, \mathcal{H}\} \ \dots \ \text{Poisson bracket}} + \frac{\partial A}{\partial t}$$

Then, in this new notation

$$\boxed{\frac{dA}{dt} = \{A, \mathcal{H}\} + \frac{\partial A}{\partial t}} \quad \text{Generalized Hamilton's Eqn}$$

First, let's make sure this makes sense for q_i and p_i. Note trivially $q_i(\vec{q}, \vec{p}, t)$. Then

$$\frac{dq_j}{dt} = \{q_j, \mathcal{H}\} + \cancelto{0}{\frac{\partial q_j}{\partial t}}$$

$$= \sum_i \left[\cancelto{\delta_{ij}}{\frac{\partial q_j}{\partial q_i}}\frac{\partial \mathcal{H}}{\partial p_i} - \cancelto{0}{\frac{\partial q_j}{\partial p_i}}\frac{\partial \mathcal{H}}{\partial q_i} \right]$$

$\Rightarrow \dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}$. Also,

$$\frac{dp_j}{dt} = \{p_j, \mathcal{H}\} + \cancelto{0}{\frac{\partial p_j}{\partial t}}$$

$$= \sum_i \left[\cancelto{0}{\frac{\partial p_j}{\partial q_i}}\frac{\partial \mathcal{H}}{\partial p_i} - \cancelto{\delta_{ij}}{\frac{\partial p_j}{\partial p_i}}\frac{\partial \mathcal{H}}{\partial q_i} \right] = -\frac{\partial \mathcal{H}}{\partial q_j}$$

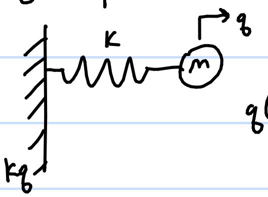
$\Rightarrow \dot{p}_j = \frac{-\partial \mathcal{H}}{\partial q_j}$

Thus, the generalized Hamilton's eqns recover the standard ones.

Next we will apply this to a probability density ρ, but first we must motivate what we mean.

ex) Probability density ρ

For clarity, consider a single particle in a harmonic potential:



$\omega = \sqrt{K/m}$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

$q(0) = q_0, \; p(0) = 0$

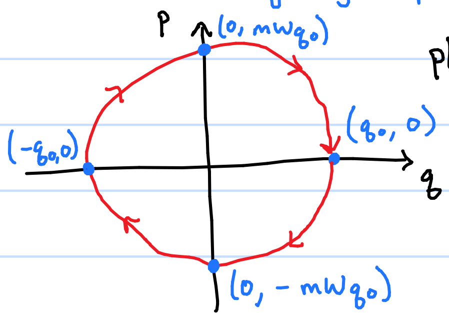
$$\Rightarrow \dot{q} = \frac{\partial \mathcal{H}}{\partial p} = p/m \;, \quad \dot{p} = \frac{-\partial \mathcal{H}}{\partial q} = -m\omega^2 q = -kq$$

$$\Rightarrow m\ddot{q} = -kq = -m\omega^2 q \quad \Rightarrow \quad q(t) = q_0 \cos(\omega t)$$

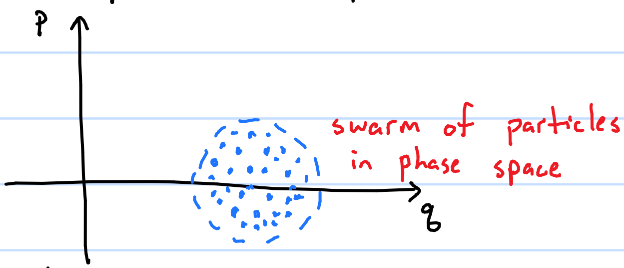
$$p(t) = -m\omega q_0 \sin(\omega t)$$

We can explicitly trace out this trajectory in phase space:

Then, as at right, we can think of the system's dynamics as a flow through phase space.

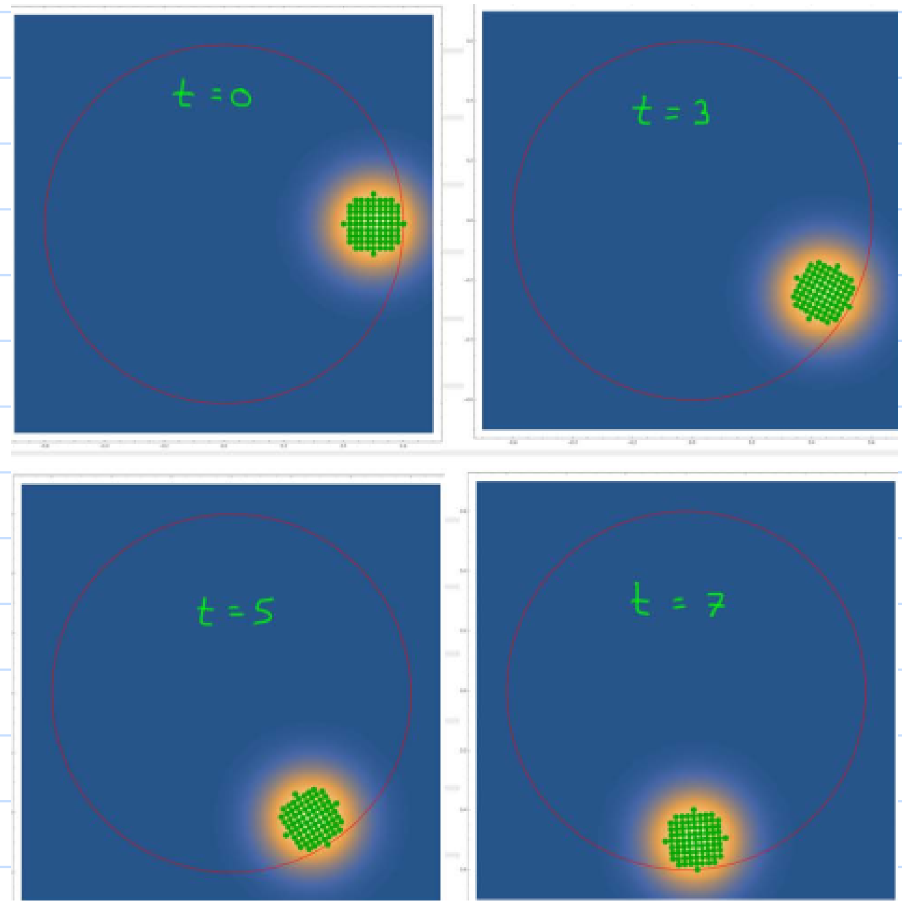


Now suppose we have a set of initial conditions $\{(q_{01}, p_{01}), (q_{02}, p_{02}), \ldots, (q_{0N}, p_{0N})\}$



What will happen to the swarm? Each blue dot will move clockwise like the single particle did. On the right, this is implemented in Mathematica.

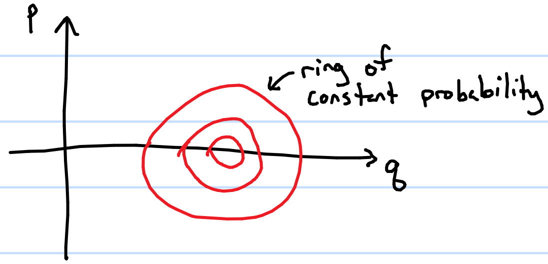
(from Mark Lusk, Colorado School of Mines)



For a harmonic oscillator, it turns out we could have used a swarm of any shape (triangle, square, etc) and the swarm will stay in the same shape but rotate around.

Next we will assume we have N identical harmonic systems. This collection is called an ensemble.

We will start each IC of the ensemble at a slightly different value according to some distribution. For example, if we choose a Gaussian distribution, in phase space our initial ensemble looks like



Let $\rho_0(q,p)$ be the probability function associated with the ICs: $\Rightarrow \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, \rho_0(q,p) = 1$

Conservation of probability

How does $\rho(q,p,t)$ evolve in time, with $\rho_0(q,p)$ as an initial condition? (This is a standard question in Hamiltonian mechs, statistical mechanics, and quantum mechanics.)

$\rho(q,p,t)$ ... observable quantity, so

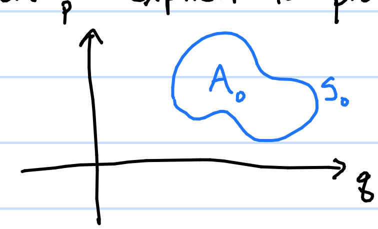
$$\frac{d\rho}{dt} = \{\rho, \mathcal{H}\} + \frac{\partial\rho}{\partial t}.$$

Our probability density satisfies a continuity equation:

$$\frac{d\rho}{dt} = \underbrace{\nabla\rho\cdot\begin{bmatrix}\dot q\\ \dot p\end{bmatrix}}_{\text{prob density flowing out}} + \underbrace{\frac{\partial\rho}{\partial t}}_{\text{prob density increasing}} = 0$$

"Conservation of probability"

If we treat ρ as a fluid, this statement says that any probability leaving a region $\left(\nabla\rho\cdot\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} > 0\right)$ is balanced by that region losing some probability density $\left(\frac{\partial\rho}{\partial t} < 0\right)$, i.e. that probability is a conserved quantity. We can be more explicit to prove this. Consider



$$\frac{d}{dt}\int_{A_0} dA\,\rho,$$

which describes how the probability you find a particle in an area A_0 changes. Let $\vec v = \begin{bmatrix}\dot q\\ \dot p\end{bmatrix}$ be the "velocity" of a particle in phase space, and let $\vec J = \rho\vec v$ where $\vec J$ is the particle "current" in phase space. Since probability cannot be created or destroyed, any change inside an area A_0 must be due to probability current $\vec J$ entering or exiting:

$$\frac{d}{dt}\int_{A_0} dA\,\rho = -\oint_{S_0} d\vec r\cdot\vec J \overset{\text{divergence theorem}}{=} -\int_{A} dA\,\nabla\cdot\vec J$$

$\frac{\partial}{\partial t}$, not $\frac{d}{dt}$ by Leibniz rule

Localizing, $$\frac{\partial\rho}{\partial t} = -\left(\frac{\partial}{\partial q}(\rho\dot q) + \frac{\partial}{\partial p}(\rho\dot p)\right)$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\dot q + \underbrace{\rho\underbrace{\frac{\partial\dot q}{\partial q}}_{=\frac{\partial}{\partial q}\left(\frac{\partial\mathcal{H}}{\partial p}\right)} + \frac{\partial\rho}{\partial p}\dot p + \rho\underbrace{\frac{\partial\dot p}{\partial p}}_{=\frac{\partial}{\partial p}\left(-\frac{\partial\mathcal{H}}{\partial q}\right)}}_{\text{cancel}} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\dot q + \frac{\partial\rho}{\partial p}\dot p = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\rho = 0}$$

From the generalized Hamilton's equations recall

$$\frac{d\rho}{dt} = \{\rho, \mathcal{H}\} + \frac{\partial \rho}{\partial t}.$$

With $\frac{d\rho}{dt} = 0$, we find

$$\boxed{\frac{\partial \rho}{\partial t} = -\{\rho, \mathcal{H}\}, \quad \begin{matrix} \rho = \rho(q, p, t) \\ \rho(0) = \rho_0(q, p) \end{matrix}}$$

Liouville's Equation

This equation describes how an <u>ensemble</u> will evolve in time.

What does Liouville's equation look like for a harmonic oscillator w/ $\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$?

$$\frac{\partial \rho}{\partial t} = -\{\rho, \mathcal{H}\} = -\left[\frac{\partial \rho}{\partial q}\underbrace{\frac{\partial \mathcal{H}}{\partial p}}_{=p/m} - \underbrace{\frac{\partial \mathcal{H}}{\partial q}}_{=m\omega^2 q}\frac{\partial \rho}{\partial p}\right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{-p}{m}\frac{\partial \rho}{\partial q} + m\omega^2 q \frac{\partial \rho}{\partial p},$$

which can be transformed via $p \to m\omega p$, $q \to q$, $t \to t/\omega$ "nondimensionalized" into

$$\boxed{\frac{\partial \rho}{\partial t} = -p\frac{\partial \rho}{\partial q} + q\frac{\partial \rho}{\partial p}}$$

$$\rho(q, p, t=0) = \rho_0(q, p) \} \text{ IC and}$$
$$\rho(\pm\infty, \pm\infty, 0) = 0 \} \text{ boundary condition}$$

This is a partial differential equation.

It turns out this PDE may be solved explicitly but we won't do that here. The main point is that we <u>can</u> solve for the evolution of the probability density w/ Liouville's Eqn.

Once we know $\rho(\vec{q}, \vec{p}, t)$, we can track the <u>expected value</u> of any observable, e.g. kinetic energy, potential energy, temperature, pressure. Call the quantity of interest f. Then

$$\underbrace{\langle f \rangle}(t) = \int d\vec{q}\, d\vec{p}\, f(\vec{q}, \vec{p}, t)\, \rho(\vec{q}, \vec{p}, t)$$

$\langle \cdot \rangle$ ... expected value

... weighted average of f

We can connect this now to statistical mechanics. Suppose we choose $\rho(\vec{q}, \vec{p})$ to be the probability a particle at $(\vec{q}, \vec{p})$ has energy E, so that

$$\rho_E(\vec{q}, \vec{p}, 0) = \begin{cases} 1, & E < \mathcal{H}(\vec{q}, \vec{p}, 0) < E + \Delta E \\ 0, & \text{else} \end{cases}$$

Then we could watch this distribution (= this volume of phase space) evolve in time, and ask how average properties e.g. pressure change in time. This ρ_E is the microcanonical ensemble in stat mech.

If instead we choose $\rho_T(\vec{q}, \vec{p}, 0) = e^{-\mathcal{H}(\vec{q}, \vec{p})/k_B T}$, we are describing the canonical ensemble.

These two ensembles form the launch point for statistical mechanics + thermodynamics.

Lastly, we will draw analogies between classical and quantum mechanics. Note:

$$\{q,q\} = \frac{\partial q}{\partial q}\frac{\partial q}{\partial p} - \frac{\partial q}{\partial q}\frac{\partial q}{\partial p} = 0.$$

$$\{p,p\} = 0$$

$$\{q,p\} = \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial p}{\partial q}\frac{\partial q}{\partial p} = 1 = -\{p,q\}$$

Further, it is true $\{q_i, q_j\} = \{p_i, p_j\} = 0$,
$\{q_i, p_j\} = \delta_{ij}$

Lastly, $\dfrac{dA}{dt} = \{A, H\} + \dfrac{\partial A}{\partial t}$.

In quantum mechanics there is an operator called the commutator $[\cdot]$, with $[A, B] = AB - BA$. In QM everything is a matrix, so $[A,B] \neq 0$ in general.

In QM there are position and momentum operators, $\hat{X}$ and $\hat{P}$. It turns out $[\hat{X}, \hat{P}] = i\hbar$; this causes the uncertainty relation. Further, it is true $[\hat{X}, \hat{X}] = [\hat{P}, \hat{P}] = 0$. Lastly, for any quantum observable $\hat{A}$, $\dfrac{d\hat{A}}{dt} = \dfrac{-i}{\hbar}[\hat{A}, \hat{H}] + \dfrac{\partial \hat{A}}{\partial t}$. This is called the Heisenburg equation of motion.

Note the similarities!

$\{x,x\} = \{p,p\} = 0$	$[\hat{X}, \hat{X}] = [\hat{P}, \hat{P}] = 0$
$\{x,p\} = 1$	$[\hat{X}, \hat{P}] = i\hbar$
$\dfrac{dA}{dt} = \{A, \mathcal{H}\} + \dfrac{\partial A}{\partial t}$	$\dfrac{d\hat{A}}{dt} = \dfrac{-i}{\hbar}[\hat{A}, \hat{\mathcal{H}}] + \dfrac{\partial \hat{A}}{\partial t}$

Deep down these similarities occur because CM and QM are described by similar algebraic structures (wiki symplectic group). This shouldn't be a huge surprise, since ultimately CM is an approximation of QM, and so the math that characterizes the two ought to be similar. At the end of the day, what this means is that at a rough level you can think about QM with your CM intuition. Indeed, this is what Paul Dirac when he was discovering QM! He leaned heavily on the similarity between the Poisson bracket + the commutator.

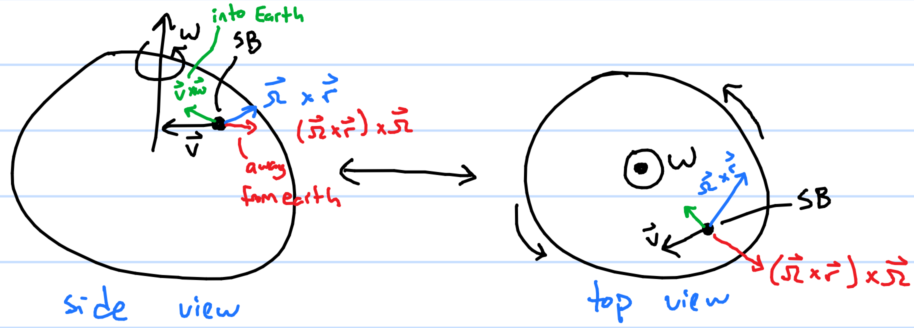
Phys 104 Final Review

The midterm will cover Taylor chapters 6, 7, 8, 9, 10, 11, and 13. The midterm will be more heavily weighted towards material we covered after the midterm. To study, I recommend reading my lecture notes, HW solutions, the midterm solns, discussion section WUS, and the textbook.

Concepts you should be familiar with:

- Noninertial reference frames. Time derivatives differ in inertial/noninertial frames $\left(\frac{d\vec{Q}}{dt}\right)_{S_0} = \left(\frac{d\vec{Q}}{dt}\right)_S + \vec{\omega}\times\vec{Q}$. N2L in rotating frame $m\ddot{\vec{r}} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}$. Coriolis force $\vec{F}_{cor} = 2m\dot{\vec{r}}\times\vec{\omega}$. Centrifugal force $\vec{F}_{cf} = m(\vec{\omega}\times\vec{r})\times\vec{\omega}$. Foucalt's pendulum.
- Rigid body rotations. Center of mass and relative coords $\vec{r}_\alpha = \vec{R}_\alpha + \vec{r}\,'_\alpha$. Kinetic energy decomposes into CM and relative terms; focus on $T_{rot} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$. Rotating about $\hat{z}$, $\vec{L} = I_{xz}\omega_x\hat{x} + I_{yz}\omega_y\hat{y} + I_{zz}\omega_z\hat{z}$, and in general $\vec{L} = I\vec{\omega}$. e.g. $I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$, $I_{zz} = \sum_\alpha m_\alpha(x_\alpha^2 + y_\alpha^2)$. Principle axes $\vec{L} = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$. Euler equations $\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \Gamma_1$ and etc. Spinning top axial symmetry $\Rightarrow \lambda_1 = \lambda_2$. Euler angles (θ, ϕ, ψ). Kinetic energy of spinning top $T = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$.
- Coupled oscillators. Everything is a harmonic oscillator. Matrix form eqn $M\ddot{\vec{x}} = -K\vec{x}$. Guess ansatz $\vec{a}e^{i\omega t}$ $\Rightarrow -M\omega^2\vec{a} = -K\vec{a} \Rightarrow (K - \omega^2 M)\vec{a} = 0 \Leftrightarrow \det(K - \omega^2 M) = 0$. Find normal frequencies ω (eigenvalue) with $\det(\cdot)$. Plug in normal frequencies to find normal mode $\vec{a}$ (eigenvector). If N masses, then N normal modes.
- Hamiltonian mechanics. Trade $(q, \dot{q})$ for phase space (q, p). Introduce Hamiltonian $\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}(\vec{q}, \dot{\vec{q}}, t)$. Hamilton's equations $\dot{q}_i = \frac{\partial\mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial q_i}$. 2N 1^{st} order ODEs vs. N 2^{nd} order ODEs. Evolution of any observable A w/ gen'lized Hamilton's Eqns $\frac{dA}{dt} = \{A, \mathcal{H}\} + \frac{\partial A}{\partial t}$, poisson bracket $\{A, \mathcal{H}\} = \sum_i\left[\frac{\partial A}{\partial q_i}\frac{\partial\mathcal{H}}{\partial p_i} - \frac{\partial\mathcal{H}}{\partial q_i}\frac{\partial A}{\partial p_i}\right]$. Similar algebraic structure to quantum mechanics. Apply gen'lized eqns to probability density ρ $\Rightarrow$ Liouville's eqn, $\frac{\partial\rho}{\partial t} = -\{\rho, \mathcal{H}\}$. Gives PDE to solve that governs evolution of an ensemble. Jumping off point for statistical mechanics.

ex) Compute the Coriolis and centrifugal force on a rocket launched west from Santa Barbara

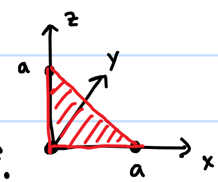


$F_{cf} = m(\vec\Omega\times\vec r)\times\vec\Omega$ = "up", away from earth

⇒ F_{cf} changes magnitude of observed gravity

$F_{cor} = 2m\dot{\vec r}\times\vec\Omega$ = "down", into earth. This is why rockets are usually launched eastward!

ex) Compute the center of mass and the inertia tensor for a flat triangle of surface density σ with vertices at $(a,0,0)$, $(0,0,0)$, $(0,0,a)$



Center of mass $R = \sum_\alpha \frac{m_\alpha r_\alpha}{M} = \frac{1}{M}\int dm\,\vec r$.

$M = \sigma\left(\frac{a^2}{2}\right) = \frac{\sigma a^2}{2}$, $dm = \sigma dA = \sigma\,dx\,dy$

$R_x = \frac{2}{\sigma a^2}\int_0^a dx\int_0^{a-x} dz\,\sigma x = \frac{2}{a^2}\int_0^a dx(ax-x^2)$

$= \frac{2}{a^2}\left[\frac{a^3}{2}-\frac{a^3}{3}\right] = \frac{2}{a^2}\frac{a^3}{6} = \frac{1}{3}a$

$R_y = \frac{2}{a^2}\int_0^a dx\int_0^{a-x} dz\; y = 0$

$R_z = \frac{2}{a^2}\int_0^a dx\int_0^{a-x} dz\; z = \frac{2}{a^2}\int_0^a dx\,\frac{(a-x)^2}{2} = \frac{1}{a^2}\int_0^a x^2-2ax+a^2$

$= \frac{1}{a^2}\left[\frac{a^3}{3}-a^3+a^3\right] = \frac{1}{3}a$ ⇒ $\vec R = \frac{1}{3}a\hat x + \frac{1}{3}a\hat z$

$I_{xx} = \sum_\alpha m_\alpha(y_\alpha^2+z_\alpha^2) = \int dm\,(y^2+z^2)$.

$dm = \sigma dA$

⇒ $I_{xx} = \sigma\int_0^a dx\int_0^{a-x} dz\,(\cancel{y^2}^{\,0}+z^2) = \sigma\int_0^a dx\,\frac{(a-x)^3}{3} = \frac{-\sigma}{12}\left[(a-x)^4\right]_0^a$

$= \frac{\sigma a^4}{12}$

⇒ I_{zz} by symmetry.

$I_{yy} = \sigma\int_0^a dx\int_0^{a-x} dz\,(x^2+z^2) = \sigma\int_0^a dx\left[\frac{2}{3}(a-x)^3\right] = \frac{\sigma a^4}{6}$

$I_{xz} = -\sigma\int_0^a dx\int_0^{a-x} dz\,xz = -\sigma\int_0^a dx\,x\,\frac{(a-x)^2}{2} = \frac{-\sigma}{2}\int_0^a dx\left[x^3-2ax^2+a^2x\right]$

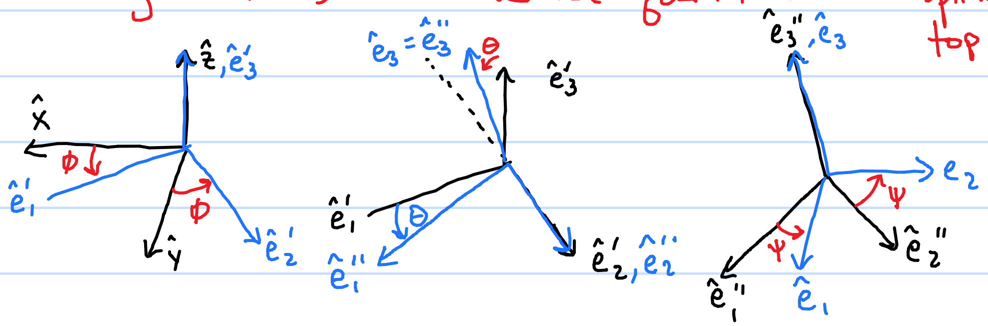
$= \frac{-\sigma}{2}\left[\frac{a^4}{4}-\frac{2}{3}a^4+\frac{a^4}{2}\right] = \frac{-\sigma a^4}{2}\left[\frac{3}{12}-\frac{8}{12}+\frac{6}{12}\right] = \frac{-\sigma a^4}{24}$

$I_{yz} = -\sigma\int_0^a dx\int_0^{a-x} dz\,\cancel{y}^{\,0}z = 0 = I_{xy}$.

⇒ $I = \frac{\sigma a^4}{24}\begin{bmatrix}2&0&-1\\0&4&0\\-1&0&2\end{bmatrix}$

Next we will cover 2 examples that were in my lecture notes but that we didn't get to in class: triple coupled pendulums, and conservation laws of a spinning top.

Ex) Symmetries + conserved quantities of a spinning top



Recall the Euler angles.

Then, we may rewrite $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2'' + \dot{\psi}\hat{e}_3$ as

$$(*) \quad \vec{\omega} = -\dot{\phi}\sin\theta\, \hat{e}_1'' + \dot{\theta}\hat{e}_2'' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3$$

Thus, when $\lambda_1 = \lambda_2$ we have written $\vec{\omega}$ in terms of its principal axes, so

$$(*) \quad \vec{L} = -\lambda_1 \dot{\phi}_1 \sin\theta\, \hat{e}_1'' + \lambda_1 \dot{\theta}\hat{e}_2'' + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\hat{e}_3,$$

and since kinetic energy is $T = \frac{1}{2}(\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2)$,

$$(*) \quad T = \frac{1}{2}\lambda_1\left[\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right] + \frac{1}{2}\lambda_3\left[\dot{\psi} + \dot{\phi}\cos\theta\right]^2$$

Note this is independent of our choice of $\hat{e}_1'', \hat{e}_2'', \hat{e}_3$.

Now we can do Lagrangian mechanics for a spinning top (with $\lambda_1 = \lambda_2$). The potential energy is $U = mgR\cos\theta$. Then,

$$\mathcal{L} = \frac{1}{2}\lambda_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - M_gR\cos\theta$$

$q_i = \theta, \phi, \psi$. Notice ϕ and ψ are ignorable.

The E-L eqns are:

θ eqn: $\lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2\sin\theta\cos\theta - \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + M_gR\sin\theta$

ϕ eqn: $p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{const}$

ψ eqn: $p_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \underbrace{\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)}_{= L_3} = \text{const}$

Notice p_ψ is simply the angular momentum in the $\hat{e}_3$ direction, which is conserved.

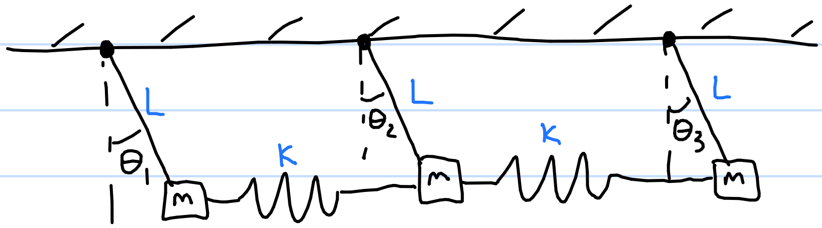
Also, we can use $\hat{z} = \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1''$ to see

$$L_z = \underbrace{\lambda_3\cos\theta(\dot{\psi} + \dot{\phi}\cos\theta) + \lambda_1\dot{\phi}\sin^2\theta}_{= L_z}$$

Notice p_ϕ is simply the angular momentum in the z-direction, which is conserved since there is no torque in the z-direction. (We saw this in Euler's equations, too.)

Therefore we may fully understand rigid body rotation in terms of Lagrangian mechanics and conserved quantities.

Ex) 3 coupled pendulums



Consider $\theta_1, \theta_2, \theta_3$ small. Then the Lagrangian is:

$$\mathcal{L} = \overbrace{\frac{1}{2} mL^2\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2\right)}^{T} + \overbrace{mgL\left(\cos\theta_1 + \cos\theta_2 + \cos\theta_3\right)}^{U_{grav}}$$
$$\underbrace{- \frac{1}{2} k\left[(L\sin\theta_2 - L\sin\theta_1)^2 + (L\sin\theta_3 - L\sin\theta_2)^2\right]}_{U_{spring}}$$
$$\approx \frac{1}{2} mL^2\left(\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2\right) + 3mgL - \frac{mgL}{2}\left(\theta_1^2 + \theta_2^2 + \theta_3^2\right)$$
$$- \frac{1}{2} kL^2\left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2\right]$$

$$\Rightarrow \quad mL^2\ddot\theta_1 = -mgL\theta_1 + kL^2(\theta_2 - \theta_1)$$
$$mL^2\ddot\theta_2 = -mgL\theta_2 - kL^2(\theta_2 - \theta_1) + kL^2(\theta_3 - \theta_2)$$
$$mL^2\ddot\theta_3 = -mgL\theta_3 - kL^2(\theta_3 - \theta_2)$$

$$\Rightarrow \begin{bmatrix} mL^2 & 0 & 0 \\ 0 & mL^2 & 0 \\ 0 & 0 & mL^2 \end{bmatrix}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{bmatrix} = -\begin{bmatrix} mgL + kL^2 & -kL^2 & 0 \\ -kL^2 & mgL + kL^2 + kL^2 & -kL^2 \\ 0 & -kL^2 & mgL + kL^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$\Leftrightarrow \quad M\ddot{\vec\theta} = -K\vec\theta.$$

Guess $\vec\theta(t) = \mathrm{Re}[\vec z(t)] = \mathrm{Re}[\vec a e^{i\omega t}]$

$$\Rightarrow -\omega^2 M\vec z = -K\vec z \Rightarrow (K - \omega^2 M)\vec z = 0$$

$$\Rightarrow (K - \omega^2 M)\vec a = 0 \Rightarrow \text{Ansatz}$$

works when $\det(K - \omega^2 M) = 0$.

$$K - \omega^2 M = \begin{bmatrix} mgL + kL^2 - \omega^2 mL^2 & -kL^2 & 0 \\ -kL^2 & mgL + 2kL^2 - \omega^2 mL^2 & -kL^2 \\ 0 & -kL^2 & mgL + kL^2 - \omega^2 mL^2 \end{bmatrix}$$

$$\det(K - \omega^2 M) = (mgL + kL^2 - \omega^2 mL^2)^2(mgL + 2kL^2 - \omega^2 mL^2)$$
$$- (mgL + kL^2 - \omega^2 mL^2)(kL^2)^2$$
$$+ kL^2(-kL^2)(mgL + kL^2 - \omega^2 mL^2)$$

Let $a = mgL + kL^2 - \omega^2 mL^2$, $b = kL^2$

$$\Rightarrow \det(K - \omega^2 M) = a^2(a+b) - ab^2 - b^2 a$$
$$= a\left[a^2 + ab - 2b^2\right] = a(a+2b)(a-b)$$
$$0 = \underbrace{(mgL + kL^2 - \omega^2 mL^2)}_{\omega_1}\underbrace{(mgL + 3kL^2 - \omega^2 mL^2)}_{\omega_2}\underbrace{(mgL - \omega^2 mL^2)}_{\omega_0}$$

$$\Rightarrow \omega_0 = \sqrt{g/L}$$
$$\omega_1 = \sqrt{\frac{g}{L} + \frac{k}{m}}$$
$$\omega_2 = \sqrt{\frac{g}{L} + 3\frac{k}{m}}$$

What are the eigenmodes?

$$\left[K^2 - \omega^2 M\right]_{\omega = \omega_0} = \begin{bmatrix} kL^2 & -kL^2 & 0 \\ -kL^2 & 2kL^2 & -kL^2 \\ 0 & -kL^2 & kL^2 \end{bmatrix}$$

Need $(K^2 - \omega_0^2 M)\vec a_{(1)} = 0 \Rightarrow \vec a_{(1)} = \begin{bmatrix} A \\ A \\ A \end{bmatrix} e^{-i\delta_1}$

$$\Rightarrow \theta_1(t) = \theta_2(t) = \theta_3(t) = A\cos(\omega_0 t - \delta_1)$$

in phase!

$$K-\omega^2 M = \begin{bmatrix} mgL+kL^2-\omega^2 mL^2 & -kL^2 & 0 \\ -kL^2 & mgL+2kL^2-\omega^2 mL^2 & -kL^2 \\ 0 & -kL^2 & mgL+kL^2-\omega^2 mL^2 \end{bmatrix}$$

Next, $\omega_1 = \sqrt{\frac{g}{L}+\frac{k}{m}}$

$$\Rightarrow K-\omega_1^2 M = \begin{bmatrix} 0 & -kL^2 & 0 \\ -kL^2 & kL^2 & -kL^2 \\ 0 & -kL^2 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{a}_{(2)} = \begin{bmatrix} A \\ 0 \\ -A \end{bmatrix} e^{-i\delta_2} \Rightarrow \begin{aligned} \theta_1(t) &= A\cos(\omega_1 t-\delta_1) \\ \theta_2(t) &= 0 \\ \theta_3(t) &= -A\cos(\omega_1 t-\delta_1) \end{aligned}$$

Lastly, $\omega_2 = \sqrt{\frac{g}{L}+\frac{3k}{m}}$

$$K-\omega_2^2 M = \begin{bmatrix} -2kL^2 & -kL^2 & 0 \\ -kL^2 & -kL^2 & -kL^2 \\ 0 & -kL^2 & -2kL^2 \end{bmatrix}$$

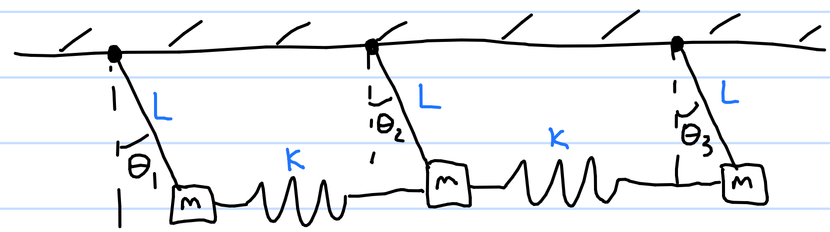
$$[K-\omega_2^2 M]\vec{a}_{(3)} = 0$$

$$\Rightarrow \vec{a}_{(3)} = \begin{bmatrix} A \\ -2A \\ A \end{bmatrix} e^{-i\delta_3}$$

$$\Rightarrow \begin{aligned} \theta_1(t) &= A\cos(\omega_2 t-\delta_3) \\ \theta_2(t) &= -2A\cos(\omega_2 t-\delta_3) \\ \theta_3(t) &= A\cos(\omega_2 t-\delta_3) \end{aligned}$$

In general, θ_1, θ_2, and θ_3 may be any linear combination of these solns (i.e. may have components of $a\cos(\omega_0 t) + b\cos(\omega_1 t) + c\cos(\omega_2 t)$)

Physically, these modes mean:



ω_0: → → →

ω_1: → 0 ←

ω_2: → ← →